Cosets, Lagrange's Theorem and factor groups

This applet can be used to display left or right cosets of subgroups of various groups of small order, and thereby to illustrate Lagrange's Theorem and the idea of a factor group. It may be used

- to demonstrate the definition and calculation of left or right cosets;
- to demonstrate the partition of a finite group into the distinct left or right cosets of a given subgroup, with all cosets being of the same size;
- to demonstrate the fact that if $b \in aH$ then $bH = aH$;
- for a normal subgroup, to display the Cayley table, arranged by cosets, and thus to display the Cayley table of the factor group;
- to compare the Cayley table of a factor group with that of a familiar group;
- for abnormal subgroups, to show that the product, as subsets, of two left cosets need not be a left coset.

The groups included are the symmetric group S_3, Klein's 4-group, dihedral groups of orders $6, 8$ and 12, the quaternionic group of order 8, cyclic groups C_i of orders $2, 3, 4, 6$ and 8, and direct products $C_2 \times C_4$ and $C_2 \times C_2 \times C_2$.

Navigation

- The initial display shows three dropdown menus: **Group**, **Subgroup** and **Second group**. Use the first of these to select a group, from a list of thirteen, and the second to select a subgroup. The Cayley table of the group will be displayed, with the elements of the subgroup highlighted in the labelling row and column.
- To compute a left coset of the chosen subgroup, click on an element in the labelling column at the side of the table. The elements will be highlighted in the appropriate row of the table. For right cosets, use the labelling row at the top.
- When all the left cosets (*resp* right cosets) have been displayed, the button **Show left cosets** (*resp* **Show right cosets**)will rearrange the Cayley table by coset.
- If the selected subgroup was normal, the Cayley table should reveal, in blocks, the Cayley table of the factor group. This can be compared with that of a familiar group using the third dropdown menu.
- If the selected subgroup is not normal, it will be clear that the set of left cosets is not closed under multiplication of subsets of the group and that the block pattern in 5 does not occur.
- The **Reset** button restores the original screen.

Examples A Investigating the left cosets of the subgroup $H := \langle r_2 \rangle$ in the dihedral group D_4 of order 8 (see configurability below re names of elements and groups).

1. From the first two dropdown menus, select D_4 and $\langle r_2 \rangle$.
2. In the labelling column at the left, click on e; the chosen subgroup appears as the left coset. Click on r_2 to obtain the same left coset, listed in a different order.
3. In the labelling column, click on r_1 to obtain, in a different colour, a second coset with two elements. Click on r_3 to obtain the same coset again. Repeat with s_1, s_3 and s_2, s_4 to obtain four distinct cosets.
4. Click on **Show left cosets** to rearrange the Cayley table by left coset, emphasising the partition into four left cosets, each with two elements.
5. Observe the block pattern in the Cayley table and select Klein's four group K from the third dropdown menu for comparison.

B Investigating the left cosets of the subgroup $H := \langle s_2 \rangle$ in the dihedral group D_4 of order 8 (see configurability below re names of elements and groups).

1. From the first two dropdown menus, select D_4 and $\langle s_2 \rangle$.
2. In the labelling column at the left, click on e; the chosen subgroup appears as the left coset. Click on s_2 to obtain the same left coset, listed in a different order.
3. In the labelling column, click on r_1 to obtain, in a different colour, a second coset with two elements. Click on s_3 to obtain the same coset again. Repeat with r_2, s_4 and r_3, s_1 to obtain four distinct cosets.
4. Click on **Show left cosets** to rearrange the Cayley table by left coset, emphasising the partition into four left cosets, each with two elements.
5. Observe that, unlike the previous example, there is no block pattern.

Configurability From the html file , the following can be configured:

1. The preferred names for the groups and their elements are set using the following parameters:
   C2, C3, C4, C6, C8 : cyclic groups of the indicated order;
   C4XC2, C2XC2XC2: the indicated direct products;
   K: Klein's 4-group;
   S3: the symmetric group of degree 3;
   D3, D4, D6: dihedral groups of order $6, 8, 12$;
   Q: the quaternion group of order 8.
   The value of each of these parameters is a finite sequence consisting of the preferred group name, followed by the names for the elements, beginning with the neutral element and closing with a comma. For cyclic groups, elements are listed in order of powers of a generator, for example,

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<param name="C4" value="C_4,e,a,a^2,a^3,">
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and for dihedral groups, represented as symmetries of regular polygons, the elements are listed with the rotations first, in anticlockwise order of angle, followed by reflections in anticlockwise order of axes of symmetry. The order for S_3 is as shown in the default html file, where permutations are given in cycle form and composition is on the left. The order for the quaternion group Q is as in the default, where the notation is standard, and for Klein's 4-group, the order of the names of the three non-identity elements is immaterial. For the two direct products, the order should be clear from the default.

2. Colours used for background, foreground (text) and highlighted text (these can also be set within the applet); the parameters "background", "foreground" and "highlight" can each be set to any of b(blue), r(red), g(green), y(yellow), c(cyan), p(pink), w(white), 0(black), n(navy).

3. The font size (this can also be set within the applet but does not affect the font size within tables); the parameter "size" can be set to any of 1(normal), 2(large), 3(huge).