

PRIME SPECTRA OF AMBISKEW POLYNOMIAL RINGS

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ABSTRACT. We determine criteria for the prime spectrum of an ambiskew polynomial algebra R over an algebraically closed field \mathbb{K} to be akin to those of two of the principal examples of such an algebra, namely the universal enveloping algebra $U(\mathfrak{sl}_2)$ (in characteristic 0) and its quantization $U_q(\mathfrak{sl}_2)$ (when q is not a root of unity). More precisely, we aim to determine when the prime spectrum of R consists of 0, the ideals $(z - \lambda)R$ for some central element z of R and all $\lambda \in \mathbb{K}$, and, for some positive integer d and each positive integer m , d height two prime ideals P for which R/P has Goldie rank m .

1. INTRODUCTION

The results of this paper are applicable to the determination of the prime ideals of ambiskew polynomial algebras and generalized Weyl algebras. For readers unfamiliar with these algebras, details appear at the end of this introduction. The main results of [12] are simplicity criteria for an ambiskew polynomial algebra R over a field \mathbb{K} and, in cases where R is not itself simple, certain localizations and factors of R including generalized Weyl algebras. Such results are applicable to the analysis of the prime spectrum of an ambiskew polynomial ring or of any ring which has an ambiskew polynomial ring as a localization. Our aim is to prove results that can prove that the prime spectrum of a given algebra R over an algebraically closed field \mathbb{K} meets the following description (*): 0 is a prime ideal, there exists $z \in Z(R)$ (the centre of R) such that the height one prime ideals have the form $(z - \lambda)R$, $\lambda \in \mathbb{K}$, $(z - \lambda)R$ is maximal for all but countably many values of λ and there is a positive integer d such that, for each $m \geq 1$, R has d height two prime ideals P for which R/P has Goldie rank m . It is well-known that the prime spectra of the universal enveloping algebra $U(\mathfrak{sl}_2)$ (in characteristic 0) and the universal quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ (when q is not a root of unity) fit the description (*) with $d = 1$ and 2 respectively. These two algebras are among the main examples of ambiskew polynomial rings. They are well-understood and will serve to illustrate our results. The new application will be to certain ambiskew polynomial rings over coordinate rings of quantum tori which arise, as localizations, in our analysis of connected quantized Weyl algebras [6].

The first step in establishing (*) for a domain is to identify an appropriate central element z for which the localization of R at $\mathbb{K}[z] \setminus \{0\}$ is simple. This will be done in Section 2 using the notion of a Casimir element for an ambiskew polynomial ring. When such elements exist, they are normal but not necessarily central. [12, Theorem 4.7] is a simplicity criterion for the localization of R at the powers of z . If z is central then this localization is never

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simple and the appropriate localization for which to consider simplicity is at $\mathbb{K}[z] \setminus \{0\}$. In Proposition 2.2, we give a simplicity criterion for this localization. As the localization is central, all ideals of R extend to ideals of the localization and simplicity of the localization is equivalent to the property that every non-zero ideal R has non-zero intersection with $\mathbb{K}[z]$. Proposition 2.9 generalizes Proposition 2.2 to a situation where there is a central polynomial subalgebra $\mathbb{K}[z, c_1, \dots, c_t]$ of R for some $t \geq 0$. This general result will be applied, with $t = 1$ to show that the augmented down-up algebras of [15] have the property that every non-zero ideal has non-zero intersection with the centre which, for these algebras, is a polynomial algebra in two indeterminates.

Having completed the first step, we proceed, in Section 3, to analyse prime spectra of the factors $R/(z - \lambda)R$ for $\lambda \in \mathbb{K}$. For description (*) to hold we need all but countably many of these to be simple. These factors are generalized Weyl algebra $W(A, \alpha, u)$ in the sense of [1] and there are applicable simplicity criteria [2, 12] for $W(A, \alpha, u)$. We also need to show that the countably many exceptions each have a unique non-zero prime factor and we shall establish sufficient conditions for this to occur, giving an explicit description of the unique non-zero prime. In Section 4, a parameter m arising in that description will be shown to be the Goldie rank of $W(A, \alpha, u)/P$ for the unique prime ideal P . For $U(sl_2)$ and the quantized enveloping algebra $U_q(sl_2)$ the exceptional maximal ideals are annihilators of finite-dimensional simple modules but this is not the case for the examples over quantum tori, where the factors are infinite-dimensional.

In the remainder of the introduction, we give some reminders of the construction and properties of ambiskew polynomial rings and generalized Weyl algebras.

Definitions 1.1. Let \mathbb{K} be a field, and let A be a \mathbb{K} -algebra. For convenience, we shall assume that \mathbb{K} is algebraically closed. Let $\rho \in \mathbb{K} \setminus \{0\}$ and let v be a central element of A . Let $\alpha \in \text{Aut}_{\mathbb{K}} A$ and let $\beta = \alpha^{-1}$. Extend β to a \mathbb{K} -automorphism of $A[y; \alpha]$ by setting $\beta(y) = \rho y$. There is a β -derivation δ of $A[y; \alpha]$ such that $\delta(A) = 0$ and $\delta(y) = v$. The *ambiskew polynomial algebra* $R(A, \alpha, v, \rho)$ is the iterated skew polynomial algebra $A[y; \alpha][x; \beta, \delta]$. Thus $ya = \alpha(a)y$ and $xa = \beta(a)x$ for all $a \in A$ and $xy = \rho yx + v$.

More general versions of ambiskew polynomial algebras are considered in [12], where v need not be central and β need not be α^{-1} , and [10], where α need not be bijective, but here we consider only the case specified above.

If there is a central element $u \in A$ such that $v = u - \rho\alpha(u)$ then the element $z = xy - u = \rho(yx - \alpha(u))$ is such that $zy = \rho yz$, $zx = \rho^{-1}xz$ and $za = az$ for all $a \in A$. Hence z is normal in R , i.e. $zR = Rz$, and it is central if and only if $\rho = 1$. If such an element u exists then it is called a *splitting* element and we say that R is a *conformal ambiskew polynomial algebra*. We then refer to the element $z := xy - u = \rho(yx - \alpha(u))$ as the *Casimir element* of R . If $\rho = 1$ then u and z are not unique and, for any $\lambda \in K$, can be replaced by $u - \lambda$ and $z + \lambda$ respectively.

Let $v^{(0)} = 0$ and $v^{(m)} = \sum_{l=0}^{m-1} \rho^l \alpha^l(v)$ for $m \in \mathbb{N}$. In particular $v^{(1)} = v$. Each $v^{(m)}$ is central and it is easily checked, by induction, that, for $m \geq 0$,

$$xy^m - \rho^m y^m x = v^{(m)} y^{m-1} \quad \text{and} \quad (1)$$

$$x^m y - \rho^m y x^m = x^{m-1} v^{(m)} = \alpha^{1-m}(v^{(m)}) x^{m-1}. \quad (2)$$

If u is a splitting element in the conformal case then $v^{(m)} = u - \rho^m \alpha^m(u)$.

Definitions 1.2. If $R(A, \alpha, v, \rho)$ is a conformal ambiskew polynomial ring and $W = R/zR$ then, as a ring extension of A , W is generated by $X := x + zR$ and $Y = y + zR$ subject to the relations $Ya = \alpha(a)Y$ and $Xa = \beta(a)X$ for all $a \in A$. Thus W is a generalized Weyl algebra in the sense of [1]. We may denote W , which has a \mathbb{Z} -grading in which $W_0 = A$ and, for $i > 0$, $W_i = AY^i$ and $W_{-i} = AX^i$, as $W(A, \alpha, u)$. If A is a domain then, by the \mathbb{Z} -grading, so too is W .

It is easy to check inductively that, for all $m \geq 1$,

$$X^m Y^m = \prod_{i=0}^{m-1} \alpha^{-i}(u) \text{ and } Y^m X^m = \prod_{i=1}^m \alpha^i(u).$$

As observed in [12, Notation 5.3], the isomorphic skew Laurent polynomial rings $A[Y^{\pm 1}; \alpha]$ and $A[X^{\pm 1}; \alpha^{-1}]$ are the localizations of W at the Ore sets $\{Y^i : i \geq 1\}$ and $\{X^i : i \geq 1\}$ respectively.

2. SIMPLE CENTRAL LOCALIZATIONS

The following lemma, which in the Noetherian case is an immediate consequence of [14, 2.1.16(vi)], is a generalization of [12, Lemma 3.1].

Lemma 2.1. *Let B be a ring, let y be a regular element of R such that $\mathcal{Y} := \{y^i\}_{i \geq 1}$ is a right and left Ore set and let \mathcal{Z} be a multiplicatively closed set of central elements of R . Let $\mathcal{W} = \{y^i z : i \geq 1, z \in \mathcal{Z}\}$, which a right and left Ore set, and let $C = B_{\mathcal{W}}$ be the localization of B at \mathcal{Y} . If C is simple and I is a non-zero ideal of B then $y^s z \in I$ for some $s \geq 0$ and some $z \in \mathcal{Z}$.*

Proof. Note that $C = (B_{\mathcal{Z}})_y$. It follows easily from the centrality of \mathcal{Z} that $IB_{\mathcal{Z}}$ is an ideal of $B_{\mathcal{Z}}$. By [12, Lemma 3.1], $y^s \in IB_{\mathcal{Z}}$ for some $s \geq 0$. By [14, 2.1.16(iv)], $y^s z \in I$ for some $z \in \mathcal{Z}$. \square

Proposition 2.2. *Let R be a conformal ambiskew polynomial ring of the form $R(A, \alpha, v, 1)$ where A is a \mathbb{K} -algebra and v is a central regular non-unit. Let u be a splitting element and $z = xy - u$ be the corresponding Casimir element and let \mathcal{Z} be the multiplicatively closed set of central elements $\mathbb{K}[z] \setminus \{0\}$. Suppose that $A[y^{\pm 1}; \alpha]$ is simple and that $Z(A[y^{\pm 1}; \alpha]) = \mathbb{K}$. Then $R_{\mathcal{Z}}$ is simple if and only if, for all $m \geq 0$, there exists a non-zero polynomial $p(X) \in \mathbb{K}[X]$ such that $p(u) \in v^{(m)}A$.*

Proof. Suppose that for all $m \geq 0$, there exists a non-zero polynomial $p(X) \in \mathbb{K}[X]$ such that $p(u) \in v^{(m)}A$. Let $\mathcal{Y} = \{y^i\}_{i \geq 0}$ and $\mathcal{Z} = \mathbb{K}[z] \setminus \{0\}$. The argument in [7, 1.5], where A is commutative, is valid more generally and shows that \mathcal{Y} is a right and left Ore set in R and $R_{\mathcal{Y}} = A[y^{\pm 1}; \alpha][z]$. By the centrality of \mathcal{Z} , $\mathcal{W} := \{y^m p(z) : m \geq 1, p(z) \in \mathcal{Z}\}$ is a right and left Ore set in R and $R_{\mathcal{W}} = (R_{\mathcal{Y}})_{\mathcal{Z}} = (R_{\mathcal{Z}})_y$. As $A[y^{\pm 1}; \alpha]$ is simple and $Z(A[y^{\pm 1}; \alpha]) = \mathbb{K}$, it follows from [14, Lemma 9.6.9], with $V = \mathbb{K}[z]$, that $R_{\mathcal{W}}$ is simple.

Let J be a non-zero prime ideal of R and suppose that $z - \lambda \notin J$ for all $\lambda \in \mathbb{K}$. By Lemma 2.1 and the simplicity of $R_{\mathcal{W}}$, $y^m q(z) \in J$ for some $m \geq 0$ and some $q(z) \in \mathcal{Z}$. By the algebraic closure of \mathbb{K} , $q(z)$ factorizes into linear factors each of which is regular modulo J , by the centrality of z , so $y^m \in J$. We can suppose that $m \geq 0$ is minimal such that $y^m \in J$ and also that $m \geq 1$. There exists a non-zero polynomial $p(X) \in \mathbb{K}[X]$ such that $p(u) \in v^{(m)}A$. By (1), $v^{(m)}y^{m-1} \in J$ whence $v^{(m)}Ay^{m-1} \subset J$ and $p(u)y^{m-1} \in J$. As u and

xy commute, $(-z)^i = (u - xy)^i \equiv u^i \pmod{Ry}$ for $i \geq 0$ and hence $p(-z) \equiv p(u) \pmod{Ry}$. Therefore $p(-z)y^{m-1} \equiv p(u)y^{m-1} \pmod{Ry^m}$ and so, as $p(u)y^{m-1} \in J$ and $y^m \in J$, we see that $p(-z)y^{m-1} \in J$. The regularity of $p(-z)$ modulo J then gives that $y^{m-1} \in J$, contradicting the minimality of m . Thus $m = 0$ and $J = R$. Hence $z - \lambda \in J$ for some $\lambda \in \mathbb{K}$. Hence $R_{\mathcal{Z}}$ is simple.

Conversely suppose that $R_{\mathcal{Z}}$ is simple. Let $m \geq 1$. As in the proof of [12, Lemma 4.1], let J be the \mathbb{K} -subspace of R spanned by the elements of the form $x^i a y^j$ where $i > 0$ or $j \geq m$ or $a \in v^{(m)}A$. Then J is a right ideal of R and $I := \text{ann}_R(R/J)$ is an ideal of R contained in J and containing y^m . Note that $J \cap A = v^{(m)}A$. As \mathcal{Z} is central, $IR_{\mathcal{Z}}$ is a non-zero ideal of the simple ring $R_{\mathcal{Z}}$ so, by [14, Proposition 2.1.16(iv)], it follows that $p(-z) \in I$ for some non-zero polynomial $p(X) \in \mathbb{K}[X]$. Thus $p(u - xy) \in J$ and, as $x \in J$ and $uxy = xyu \in J$, it follows that $p(u) \in J \cap A = v^{(m)}A$. \square

Remark 2.3. The hypotheses in Proposition 2.2 that $Z(A[y^{\pm 1}; \alpha]) = \mathbb{K}$ and $A[y^{\pm 1}; \alpha]$ is simple can be rephrased in terms of the base ring A . Using [14, Theorem 1.8.5], it is easy to check that these conditions are equivalent to the following three conditions:

- (i) A is α -simple;
- (ii) α^n is outer for all positive integers n ;
- (iii) $\{a \in Z(A) : \alpha(a) = a\} = \mathbb{K}$.

The following lemma is applicable to show that, in the situation of Proposition 2.2, if $R_{\mathcal{Z}}$ is simple then every height one prime ideal of R is generated by an irreducible element of $\mathbb{K}[z]$.

Lemma 2.4. *Let $W = W(A, \alpha, u)$ be a generalized Weyl algebra and let I be an ideal of A such that $I = \alpha(I)$. Then IW is an ideal of W and $W/IW \simeq W(A/I, \bar{\alpha}, \bar{u})$, where $\bar{\alpha}$ is the automorphism of A/I induced by α and $\bar{u} = u + I$.*

Proof. It is routine to check that an isomorphism is given by

$$(a_i Y^i + \dots + a_0 + \dots a_{-j} X^j) + IW \mapsto (\bar{a}_i Y^i + \dots \bar{a}_0 + \bar{a}_{-j} X^j),$$

where, for $i \in \mathbb{Z}$, $\bar{a}_i = a_i + I$. \square

Corollary 2.5. *Let R be a conformal ambiskew polynomial ring of the form $R(A, \alpha, v, 1)$ where A is a \mathbb{K} -algebra and v is a central regular non-unit. Let u be a splitting element and $z = xy - u$ be the corresponding Casimir element. Let \mathcal{Z} be the multiplicatively closed set of central elements $\mathbb{K}[z] \setminus \{0\}$. Suppose that $A[y^{\pm 1}; \alpha]$ is simple, that $Z(A[y^{\pm 1}; \alpha]) = \mathbb{K}$ and that, for all $m \geq 0$, there exists a non-zero polynomial $p(X) \in \mathbb{K}[X]$ such that $p(u) \in v^{(m)}A$. Then R is a UFD (in the sense of [4]).*

Proof. Certainly R is a domain. It follows from Proposition 2.2 that if P is a height one prime ideal of R then $f \in P$ for some irreducible element $f \in \mathbb{K}[z]$. It remains to show that fR is completely prime. By [11, Corollary 2.6], R is isomorphic to the generalized Weyl algebra $W = W(B, \alpha, u)$, where $B = A[z]$ and α extends to $\mathbb{K}[z]$ with $\alpha(z) = z$. Applying Lemma 2.4 with $I = fB$, we see that R/fR is a generalized Weyl algebra over the domain B/fB and hence is a domain. \square

In the first two of the following examples it is well-known that every non-zero ideal intersects the centre non-trivially. They are included to illustrate Proposition 2.2 rather than to advance understanding of the examples.

Example 2.6. Assume that $\text{char}(\mathbb{K}) = 0$. Let A be the polynomial algebra $\mathbb{K}[t]$ and let α be the \mathbb{K} -automorphism of A such that $\alpha(t) = t + 2$. It is well-known that A is α -simple. Let $\rho = 1$ and let $u = \frac{-1}{4}(t-1)^2$, so that $v = t$. Then $R(A, \alpha, v, 1)$ is the enveloping algebra $U(\mathfrak{sl}_2)$, in which x, y and t are usually written e, f and h . In the notation of Definitions 1.1, the Casimir element z given by the formula in Definitions 1.1 is $\frac{1}{4}(\Omega + 1)$, where Ω is the usual Casimir element as, for example, in [5]. For $m \geq 1$, $v^{(m)} = m(t + m - 1)$ and $p(u) \in v^{(m)}A$ when $p(u) = u + \frac{1}{4}m^2 = \frac{1}{4}(m + t - 1)(m - t + 1)$. In accordance with Proposition 2.2, the localization of R at $\mathbb{K}[z] \setminus \{0\}$ is simple.

Example 2.7. Let $q \in \mathbb{K}$ and suppose that q is not a root of unity. Let A be the Laurent polynomial algebra $\mathbb{K}[t^{\pm 1}]$ and let α be the \mathbb{K} -automorphism of A such that $\alpha(t) = q^2 t$. Again, it is well-known that A is α -simple. Let $\rho = 1$ and let $u = -(q^{-1}t + qt^{-1})/(q - q^{-1})^2$, so that $v = (t - t^{-1})/(q - q^{-1})$. Here $R(A, \alpha, v, 1)$ is the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$, for example, see [3, Chapter I.3]. Here x, y and t are usually written E, F and K . The Casimir element z is $xy + (q^{-1}t + qt^{-1})/(q - q^{-1})^2$. For $m \geq 1$, the element $v^{(m)}$ is $((q^{2m-1} - q^{-1})t + (q^{1-2m} - q)t^{-1})/(q - q^{-1})^2$ so $v^{(m)}t$ has the form $at^2 + b$, where $a, b \in \mathbb{K}^*$. Modulo $v^{(m)}A$, $t^2 \equiv -ba^{-1}$ and $t^{-2} \equiv -ab^{-1}$ so, as u^2 has the form $ct^2 + d + et^{-2}$, for some $c, d, e \in \mathbb{K}^*$, $u^2 - \lambda \in v^{(m)}A$ for some $\lambda \in \mathbb{K}$. In accordance with Proposition 2.2, the localization of R at $\mathbb{K}[z] \setminus \{0\}$ is simple. Note that the version of $U_q(\mathfrak{sl}_2)$ considered in [7, Example 2.3] is different to the now established one considered here.

In the next example, which occurs as a localization of a connected quantized Weyl algebra in [6], A is noncommutative and the results of [8] on height one prime ideals do not apply.

Example 2.8. Let p be an odd positive integer and let $q \in \mathbb{K}^*$. Suppose that q is not a root of unity. Let A be the quantum torus with generators $z_i^{\pm 1}$, $1 \leq i \leq p$, subject to the relations $z_i z_j = q_{ij} z_j z_i$ for $1 \leq j < i \leq p$, where, for $i > j$, $q_{ij} = 1$ if i is odd or if i and j are both even, and $q_{ij} = q^{-1}$ if i is even and j is odd. Note that z_p is central in A . Let α be the \mathbb{K} -automorphism of A such that, for $1 \leq i \leq p$, $\alpha(z_i) = z_i$ if i is even and $\alpha(z_i) = q^{-1} z_i$ if i is odd. The skew Laurent polynomial ring $S = A[y^{\pm 1}; \alpha]$ is a quantum torus in $p + 1$ generators $z_i^{\pm 1}$, $1 \leq i \leq p + 1$, where $z_{p+1} = y$. It follows from [13, Proposition 1.3], that S is simple and has centre \mathbb{K} .

Let v be the central element $(1 - q)(q^{\frac{p+1}{2}} z_p^{-1} - z_p) \in A$ and observe that $v = u - \alpha(u)$, where $u = q^{\frac{p-1}{2}} z_p^{-1} + q z_p$. Thus $R := R(A, \alpha, v, 1)$ is conformal with Casimir element $\Omega := xy - u$.

Let $m \geq 1$. Then $v^{(m)} = u - \alpha^m(u) = (1 - q^m)(q^{\frac{p-1}{2}} z_p^{-1} - q^{1-m} z_p)$ so, modulo $v^{(m)}A$,

$$z_p^2 \equiv q^{\frac{p+2m-3}{2}} \quad \text{and} \quad z_p^{-2} \equiv q^{-\frac{p+2m-3}{2}}.$$

Hence

$$\begin{aligned} u^2 &= q^{p-1} z_p^{-2} + 2q^{\frac{p+1}{2}} + q^2 z_p^2 \\ &\equiv q^{\frac{p+1}{2}} (q^{-m} + 2 + q^m) \pmod{v^{(m)}A}. \end{aligned}$$

Thus $p(u) \in v^{(m)}A$ where $p(X) = X^2 - \sigma$ and $\sigma = q^{\frac{p+1}{2}} (q^{-m} + 2 + q^m)$. By Proposition 2.2 every nonzero prime ideal of R has non-zero intersection with $\mathbb{K}[\Omega]$.

The next result is a generalization of Proposition 2.2, which is the case $t = 0$, and is applicable to other algebras in which every ideal intersects the centre non-trivially.

Proposition 2.9. *Let B be a \mathbb{K} -algebra with a \mathbb{K} -automorphism α such that $B[y^{\pm 1}; \alpha]$ is simple and $Z(B[y^{\pm 1}; \alpha]) = \mathbb{K}$. Let $t \geq 0$ be an integer and let A be the polynomial algebra $B[c_1, \dots, c_t]$ in t algebraically independent commuting indeterminants. Extend α to a \mathbb{K} -automorphism of A by setting $\alpha(c_i) = c_i$ for $1 \leq i \leq t$. Let $u \in A$, let $v = u - \alpha(u)$ and, in the conformal ambiskew polynomial ring $R = R(A, \alpha, v, 1)$, let z be the Casimir element $xy - u$.*

(i) $Z(A[y^{\pm 1}; \alpha]) = \mathbb{K}[c_1, \dots, c_t]$ and $Z(R)$ is the polynomial algebra $\mathbb{K}[z, c_1, \dots, c_t]$.

(ii) Let $\mathcal{Z} = Z(R) \setminus 0$. Then the localization $R_{\mathcal{Z}}$ is simple if and only if, for all $m \geq 0$, there exists a non-zero polynomial $p(X, X_1, \dots, X_t) \in \mathbb{K}[X, X_1, \dots, X_t]$ such that $p(u, c_1, \dots, c_t) \in v^{(m)}A$.

Proof. (i) is straightforward.

(ii) We adapt the proof of Proposition 2.2 with $\mathcal{Y} = \{y^i\}_{i \geq 0}$, $R_{\mathcal{Y}} = A[y^{\pm 1}; \alpha][z] = B[y^{\pm 1}; \alpha][z, c_1, \dots, c_t]$, $\mathcal{W} = \{y^m p(z, c_1, \dots, c_t) : m \geq 1, p(z, c_1, \dots, c_t) \in \mathcal{Z}\}$ and $R_{\mathcal{W}} = (R_{\mathcal{Y}})_{\mathcal{Z}} = (R_{\mathcal{Z}})_{\mathcal{Y}}$, which is simple.

Suppose that $R_{\mathcal{Z}}$ is not simple, let $M \neq 0$ be a maximal ideal of $R_{\mathcal{Z}}$ and let $J = M \cap R$. Then $\mathcal{Z} \cap J = \emptyset$, $J \neq 0$ and, using the centrality of \mathcal{Z} , it is easy to check that J is a prime ideal of R . By Lemma 2.1 and the simplicity of $R_{\mathcal{W}}$, $y^m q(z, c_1, \dots, c_t) \in J$ for some $m \geq 0$ and some $q(z, c_1, \dots, c_t) \in \mathcal{Z}$. By the centrality of $q(z, c_1, \dots, c_t)$, $q(z, c_1, \dots, c_t)$ is regular modulo J so $y^m \in J$. We can suppose that m is minimal such that $m \geq 0$ and $y^m \in J$. As J is proper, $m \geq 1$. There exists a non-zero polynomial $p(X, X_1, \dots, X_t) \in \mathbb{K}[X, X_1, \dots, X_t]$ such that $p(u, c_1, \dots, c_t) \in v^{(m)}A$. As in the proof of Proposition 2.2, $p(u, c_1, \dots, c_t)y^{m-1} \in J$, $p(-z, c_1, \dots, c_t) \equiv p(u, c_1, \dots, c_t) \pmod{Ry}$, $p(-z, c_1, \dots, c_t)y^{m-1} \equiv p(u, c_1, \dots, c_t)y^{m-1} \pmod{Ry^m}$, $p(-z, c_1, \dots, c_t)y^{m-1} \in J$ and $y^{m-1} \in J$, contradicting the minimality of m . It follows that $R_{\mathcal{Z}}$ is simple.

Conversely suppose that $R_{\mathcal{Z}}$ is simple. Let $m \geq 1$. As in the proof of Proposition 2.2, if J denotes the \mathbb{K} -subspace of R spanned by the elements of the form $x^i a y^j$ where $i > 0$ or $j \geq m$ or $a \in v^{(m)}A$ then J is a right ideal of R and $I := \text{ann}_R(R/J)$ is an ideal of R contained in J and containing y^m . Also $J \cap A = v^{(m)}A$. As \mathcal{Z} is central, $IR_{\mathcal{Z}}$ is a non-zero ideal of the simple ring $R_{\mathcal{Z}}$ so, by [14, Proposition 2.1.16(iv)], it follows that $p(-z, c_1, \dots, c_t) \in I$ for some non-zero polynomial $p(X, X_1, \dots, X_t) \in \mathbb{K}[X, X_1, \dots, X_t]$. Thus $p(u - xy, c_1, \dots, c_t) \in J$ and, as $x \in J$ and $uxy = xyu \in J$, it follows that $p(u, c_1, \dots, c_t) \in J \cap A = v^{(m)}A$. \square

We next look at a class of algebras, introduced by Terwilliger and Worawannotai[15], to which Proposition 2.9 applies with $t = 1$.

Example 2.10. Let $A = \mathbb{K}[c, k^{\pm 1}]$, let $q \in \mathbb{K}^*$ and suppose that q is not a root of unity. Let α be the \mathbb{K} -automorphism such that $\alpha(k) = q^2 k$ and $\alpha(c) = c$. Fix a non-zero integer n and a Laurent polynomial $f(k) = \sum a_i k^i \in \mathbb{K}[k, k^{-1}]$, such that $a_n = 0$. Let $u = ck^n + f(k)$ and $v = u - \alpha(u) = (1 - q^{2n})ck^n + \sum b_i k^i$ where each $b_i = (1 - q^{2i})a_i$. In particular $b_0 = 0$. Then $R = R(A, \alpha, v, 1)$ is generated by $k^{\pm 1}, c, x$ and y subject to the relations

$$ck = kc, \quad xc = cx, \quad yc = cy, \quad (3)$$

$$kk^{-1} = 1 = k^{-1}k, \quad (4)$$

$$xk = q^{-2}kx, \quad yk = q^2ky, \quad (5)$$

$$xy - yx = (1 - q^{2n})ck^n + \sum b_i k^i. \quad (6)$$

By (6),

$$c = (1 - q^{2n})^{-1}(xy - yx - \sum b_i k^i)k^{-n}$$

so, as a generator, c is redundant. Substituting the above expression for c in the relations $xc = cx$ and $cy = yc$ gives two relations in x, y and k that are cubic in x, y . Then R is generated by $k^{\pm 1}, x$ and y subject to these two relations and

$$kk^{-1} = 1 = k^{-1}k, \quad (7)$$

$$xk = q^{-2}kx, \quad yk = q^2ky, \quad (8)$$

$$xy - yx = (1 - q^{2n})ck^n + \sum b_i k^i. \quad (9)$$

This corresponds to the presentation in [15, Definition 2.1], but the generators there are $e = q^{-t}k^s x$ and $f = y$, where $t - s = n$. Following [15], we shall refer to R as an *augmented down-up algebra*.

By the construction above R is conformal with central Casimir element $z = xy - u$ and it is readily checked that $Z(R) = \mathbb{K}[c, z]$. For $m \geq 1$,

$$v^{(m)} = (1 - q^{mn})ck^n + \sum (1 - q^{2im})a_i k^i$$

so $A/v^{(m)}A \simeq \mathbb{K}[k^{\pm 1}]$ which is an integral domain of transcendence degree 1. Hence there exists a non-zero polynomial $p(X, Y) \in \mathbb{K}[X, Y]$ such that $p(u, c) \in v^{(m)}A$. Applying 2.9, we obtain the following Proposition.

Proposition 2.11. *If R is an augmented down-up algebra then every non-zero ideal of R has non-zero intersection with $Z(R)$ and the localization of R at $Z(R) \setminus \{0\}$ is simple.*

Corollary 2.12. *An augmented down-up algebra R is a UFD (in the sense of [4]).*

Proof. The proof is essentially the same as that of Corollary 2.5. □

3. FAMILIES OF EXCEPTIONAL SIMPLE QUOTIENTS

Although the results of this section are more widely applicable, they are aimed at the case where R satisfies the hypotheses and the simplicity criterion of Proposition 2.2. Examples include Examples 2.6, 2.7 and 2.8. We continue to assume that \mathbb{K} is algebraically closed so that every height one prime ideal P of R has the form $(z - \lambda)R$ with $\lambda \in \mathbb{K}$. The factor $R/(z - \lambda)R$ is then the generalized Weyl algebra $W(A, \alpha, u - \lambda)$ and the following result from [2] is applicable. An earlier version appeared in [9], where A is commutative, and a more general version is [12, Theorem 5.4].

Theorem 3.1. *Let α be a \mathbb{K} -automorphisms A of an \mathbb{K} -algebra A , let $u \in A$ be central and let W be the generalized Weyl algebra $W(A, \alpha, u)$. Then W is simple if and only if*

- (i) A is α -simple;
- (ii) α^m is outer for all $m \geq 1$;
- (iii) u is regular;
- (iv) $uA + \alpha^m(u)A = A$ for all $m \geq 1$.

The following lemma determines those values of λ for which $R/(z - \lambda)R$ is simple in Example 2.8.

Lemma 3.2. *Suppose that q is not a root of unity. Let A , $u = q^{\frac{p-1}{2}} z_p^{-1} + \lambda + qz_p$ and α be as in Example 2.8. Let $m \in \mathbb{N}$. Then the ideal $uA + \alpha^m(u)A$ is proper if and only if $\lambda = \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$. If $\lambda = \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$ then $uA + \alpha^m(u)A$ is a maximal (and completely prime) ideal of A and $uA + \alpha^a(u)A = A$ for all $a \in \mathbb{N} \setminus \{m\}$.*

Proof. Suppose that $uA + \alpha^m(u)A$ is proper. The maximal ideals of A have the form $(z_p - \mu)A$, $\mu \in \mathbb{K}^*$ and are completely prime with factors isomorphic to quantum tori in $p-1$ indeterminates. So there exists $\mu \in \mathbb{K}^*$ such that $u \in (z_p - \mu)A$ and $\alpha^m(u)A \in (z_p - \mu)A$ and hence such that

$$q^{\frac{p-1}{2}} \mu^{-1} + \lambda + q\mu = 0 = q^m q^{\frac{p-1}{2}} \mu^{-1} + \lambda + q^{1-m} \mu.$$

Eliminating the terms that involve μ^{-1} ,

$$\lambda(q^m - 1) + (q^{m+1} - q^{1-m})\mu = 0$$

and, dividing through by $q^m - 1$, which is necessarily non-zero, $\lambda = -q^{1-m}(q^m + 1)\mu$. Hence $\lambda \neq 0$. Also

$$\begin{aligned} 0 &= q^{\frac{p-1}{2}} q^{1-m}(q^m + 1)\lambda^{-1} - \lambda + q^m(q^m + 1)^{-1}\lambda, \\ 0 &= q^{\frac{p-1}{2}} q^{1-m}(q^m + 1)^2 - \lambda^2(q^m + 1) + q^m \lambda^2, \\ 0 &= q^{\frac{p-1}{2}} q^{1-m}(q^m + 1)^2 - \lambda^2 \text{ and} \\ \lambda &= \pm q^{\frac{p-2m+1}{4}}(q^m + 1). \end{aligned}$$

Conversely, suppose that $\lambda = \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$. Then

$$\begin{aligned} u &= (z_p \pm q^{\frac{p+2m-3}{4}})(q \pm q^{\frac{p-2m+1}{4}} z_p^{-1}) \text{ and} \\ \alpha^m(u) &= (z_p \pm q^{\frac{p+2m-3}{4}})q^{1-m}(1 \pm q^{\frac{p+6m-3}{4}} z_p^{-1}). \end{aligned}$$

Thus $uA + \alpha^m(u)A \subseteq (z_p \pm q^{\frac{p+2m-3}{4}})A \neq A$. Indeed, as $q \pm q^{\frac{p-2m+1}{4}} z_p^{-1}$ and $1 \pm q^{\frac{p+6m-3}{4}} z_p^{-1}$ generate distinct maximal ideals, $uA + \alpha^m(u)A = (z_p \pm q^{\frac{p+2m-3}{4}})A$ which is a maximal ideal of A .

Finally, if $uA + \alpha^a(u)A \neq A$ then $q^{\frac{p-2a+1}{4}}(q^a + 1) = \pm \lambda = \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$ from which it follows successively that $q^{\frac{-a}{2}}(q^a + 1) = \pm q^{\frac{-m}{2}}(q^m + 1)$, $q^{\frac{a}{2}} + q^{\frac{-a}{2}} = \pm(q^{\frac{m}{2}} + q^{\frac{-m}{2}})$, $q^a + q^{-a} = (q^m + q^{-m})$ and $q^a - q^m = (q^a - q^m)q^{-a-m}$. As q is not a root of unity, this cannot happen if $a \in \mathbb{N} \setminus \{m\}$. \square

Corollary 3.3. *If R and its Casimir element Ω are as in Example 2.8 then $(\Omega - \lambda)R$ is maximal if and only if, for all $m \geq 1$, $\lambda \neq \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$.*

Proof. This is immediate from Proposition 2.9 and Lemma 3.2. \square

Lemma 3.4. *Let $W = W(A, \alpha, u)$ be a generalized Weyl algebra with u central in A . Let $j \geq 1$ be such that $uA + \alpha^j(u)A = A$. Let J be an ideal of W . If $Y^j \in J$ then $Y^{j-1} \in J$ and if $X^j \in J$ then $X^{j-1} \in J$. Consequently, if $uA + \alpha^i(u)A = A$ for $1 \leq i \leq j$ and $Y^j \in J$ or $X^j \in J$ then $J = W$.*

Proof. If $Y^j \in J$ then $uY^{j-1} = XY^j \in J$ and $\alpha^j(u)Y^{j-1} = Y^{j-1}\alpha(u) = Y^j X \in J$, whence $AY^{j-1} = (Au + A\alpha^j(u))Y^{j-1} \subseteq J$ and so $Y^{j-1} \in J$. Similarly, if $X^j \in J$ then $\alpha(u)X^{j-1} =$

$YX^j \in J$ and $\alpha^{-(j-1)}(u)X^{j-1} = X^{j-1}u = X^jY \in J$, whence $AX^{j-1} = (A\alpha^{-(j-1)}(u) + A\alpha(u))X^{j-1} \subseteq J$ and so $X^{j-1} \in J$. Repeating the argument yields the stated consequence. \square

Proposition 3.5. *Let $W = W(A, \alpha, u)$ be a generalized Weyl algebra with u central in A . Let $m \geq 1$ be such that $uA + \alpha^j(u)A = A$ for $1 \leq j < m$ but $uA + \alpha^m(u)A \neq A$. Let I be an ideal of A containing $uA + \alpha^m(u)A$. There is a \mathbb{Z} -graded ideal $J = J(I)$ of W such that, for $i \geq 0$, $J_i = I_iY^i$ and $J_{-i} = I_{-i}X^i$, where, if $i \geq m$ then $I_i = I_{-i} = A$ and, if $0 \leq i \leq m-1$ then*

$$I_i := \bigcap_{\ell=0}^{m-1-i} \alpha^{-\ell}(I) \text{ and } I_{-i} := \bigcap_{\ell=i}^{m-1} \alpha^{-\ell}(I).$$

Proof. Note that the two definitions of I_0 coincide. With J_i as above for $i \in \mathbb{Z}$, let $J = \bigoplus_{i \in \mathbb{Z}} J_i$. It is clear that $J_iA \subseteq J_i$ and $AJ_i \subseteq J_i$ for each $i \in \mathbb{Z}$. Let $i \geq 0$. Clearly $J_iY \subseteq J_{i+1}$ and $J_{-i}X \subseteq J_{-(i+1)}$. Also,

$$YJ_i \subseteq \alpha(I_i)Y^{i+1} \subseteq I_{i+1}Y^{i+1} = J_{i+1}$$

and, similarly, $XJ_{-i} \subseteq J_{-(i+1)}$. Now let $i \geq 1$. As $u \in \alpha^{-m}(I)$ and $u \in I$,

$$J_iX = I_iY^{i-1}\alpha(u) = I_i\alpha^i(u)Y^{i-1} \subseteq I_i\alpha^{i-m}(I)Y^{i-1} \subseteq I_{i-1}Y^{i-1} = J_{i-1}$$

and

$$XJ_i = \alpha^{-1}(I_i)XY^i = \alpha^{-1}(I_i)uY^{i-1} \subseteq \alpha^{-1}(I_i)Y^{i-1} \subseteq I_{i-1}Y^{i-1} = J_{i-1}.$$

Similarly, $J_{-i}Y \subseteq J_{-(i-1)}$ and $YJ_{-i} \subseteq J_{-(i-1)}$. It follows that J is a graded ideal of W . \square

Notation 3.6. For $i \geq 1$, let $d_i = \alpha(u)\alpha^2(u) \dots \alpha^i(u)$ and $e_i = \alpha^{-i}(d_i) = u\alpha^{-1}(u) \dots \alpha^{1-i}(u)$. Thus $d_i = Y^iX^i$ and $e_i = X^iY^i$, see Definitions 1.2.

Lemma 3.7. *Let W and m be as in Proposition 3.5. For $0 \leq i < m$, $d_iA + uA = A = d_iA + \alpha^m(u)A$ and $e_iA + \alpha^{-i}(u)A = A = e_iA + \alpha^{m-1}(u)A$.*

Proof. Suppose that $d_iA + uA \neq A$ and let M be a maximal ideal of A containing $d_iA + uA$. As u is central, there exists j such that $1 \leq j \leq i < m$ such that $\alpha^j(u) \in M$ and $u \in M$. This contradicts the conditions of Proposition 3.5 so $d_iA + uA = A$. Similarly $d_iA + \alpha^m(u)A = A$ and, applying α^{-i} , $e_iA + \alpha^{-i}(u)A = A = e_iA + \alpha^{m-1}(u)A$. \square

Lemma 3.8. *Let W , I and $J = J(I)$ be as in Proposition 3.5 and suppose that I is a maximal ideal of A . Then J is a maximal ideal of W .*

Proof. Let M be an ideal of W such that $J \subset M$. There exist $a_{-(m-1)}, \dots, a_0, \dots, a_{m-1} \in A$ such that $g := a_{-(m-1)}X^{m-1} + \dots + a_0 + \dots + a_{m-1}Y^{m-1} \in M$ and $a_i \notin I_i$ for at least one i with $m-1 \geq i \geq -(m-1)$.

Suppose that $0 \leq i \leq m-1$. Then there exists ℓ such that $0 \leq \ell \leq i \leq m-1$ and $\alpha^\ell(a_i) \notin I$. But $Y^\ell g Y^{m-1-i-\ell} \in M$ and its coefficient of Y^{m-1} is $\alpha^\ell(a_i)$. Replacing g by $Y^\ell g Y^{m-1-i-\ell} \in M$ and recalling that $Y^m \in J \subseteq M$, we can assume that $i = m-1$.

Similarly, if $0 \geq i \geq 1-m$ then we replace g by $X^{m-1-\ell} g X^{\ell+i}$, where $-i \leq \ell \leq m-1$ and $\alpha^\ell(a_i) \notin I$, so that the coefficient of X^{m-1} becomes $\alpha^{1-m+\ell}(a_i) \notin \alpha^{1-m}(I) = I_{1-m}$. Thus we can assume that $i = 1-m$.

This leaves the two cases $i = \pm(m-1)$. Suppose first that $i = m-1$ so that $a_{m-1} \notin I_{m-1} = I$. Let F denote the set of all elements $f \in A$ for which there exist $b_{-(m-1)}, \dots, b_{m-2} \in A$ such that $b_{-(m-1)}X^{m-1} + \dots + b_0 + \dots + b_{m-2}Y^{m-2} + fY^{m-1} \in M$. Then F is an ideal of

A , $a_{m-1} \in F \setminus I$ and $I \subseteq F$ so $F \neq I$ and $A = F$. Hence we may assume that $a_{m-1} = 1$. Consider $X^{m-1}gX^{m-1}$, which belongs both to M and to $X^{m-1}Y^{m-1}X^{m-1} + \sum_{j=m}^{3m-2} AX^j$. As $X^m \in J$, it follows that $X^{m-1}Y^{m-1}X^{m-1} \in M$, that is $e_{m-1}X^{m-1} \in M$. As u is central in A and $\alpha^{1-m}(u)X^{-1} \in \alpha^{-(m-1)}(I)X^{m-1} = I_{-(m-1)}X^{m-1} \subseteq J$ it follows from Lemma 3.7 that $X^{m-1} \in M$. By Lemma 3.4, $M = W$.

The argument if $i = 1 - m$ is similar. We may assume that $a_{-(1-m)} = 1$ and consider $Y^{m-1}gY^{m-1}$, which belongs to M , giving that $Y^{m-1}X^{m-1}Y^{m-1} = d_{m-1}Y^{m-1} \in M$, which leads us to conclude, using Lemma 3.7 and the fact that $uX^{m-1} \in J \subseteq M$, that $M = W$. This completes the proof that J is maximal. \square

Lemma 3.9. *Let W be as in Proposition 3.5, let $I = uA + \alpha^m(u)A$ and let $J = J(I)$ be as in Proposition 3.5. Any prime ideal P of W containing X^m and Y^m must contain J .*

Proof. For $i \geq 0$, let $d_i = Y^i X^i = \alpha(u)\alpha^2(u) \dots \alpha^i(u)$. Let K be an ideal of W that contains X^m and Y^m .

We claim that $d_{m-1}J \subseteq K$. For this it suffices to show that $d_{m-1}J_i \subseteq K$ for all $i \in \mathbb{Z}$. As $Y^m \in K$ and $X^m \in K$, $J_i \subseteq K$ and $J_{-i} \subseteq K$ for $i \geq m$. Let $1 \leq i < m$. Then $J_{-i} \subseteq X^i A$ so

$$d_{m-1}J_{-i} \subseteq d_{m-1}X^i A = Y^{m-1}X^{m-1}X^i A = Y^{m-1}X^m X^{i-1} A \subseteq K.$$

Finally, for $0 \leq i \leq m$, $J_i \subseteq IY^i$ so

$$d_{m-1}J_i \subseteq d_{m-1}IY^i = ud_{m-1}Y^i A + \alpha^m(u)d_{m-1}Y^i A = ud_{m-1}Y^i A + d_m Y^i A.$$

Here $d_m = Y^m X^m \in K$ and $ud_{m-1} = XY Y^{m-1} X^{m-1} = XY^m X^{m-1} \in K$ so $d_{m-1}J_i \subseteq K$. This completes the proof of the claim that $d_{m-1}J \subseteq K$.

Now suppose that K is prime and that $J \not\subseteq K$. Then, as J is an ideal and $d_{m-1}J \subseteq K$, $d_{m-1} \in K$. Note that $X^{m-1}u = X^{m-1}XY = X^m Y \in K$ so $X^{m-1}(uA + d_{m-1}A) \subseteq K$. It follows, by Lemma 3.7, that $X^{m-1} \in K$. By Lemma 3.4, $K = W$. Therefore $J \subseteq K$. \square

Theorem 3.10. *Let $W(A, \alpha, u)$ be a generalized Weyl algebra, with u central and regular in A , such that, for some fixed $m \in \mathbb{N}$:*

- (i) $Au + A\alpha^i(u) = A$ for $0 < i < m$ and for $i > m$;
- (ii) $M := Au + A\alpha^m(u)$ is a maximal ideal in A .

Then the ideal $J(M)$ is a maximal ideal of W containing both X^m and Y^m and is the unique prime ideal in W containing X^r and Y^r for any $r \in \mathbb{N}$. Moreover if A is α -simple and no power of α is inner then $J(M)$ is the unique non-zero prime ideal in W .

Proof. By Lemmas 3.8 and 3.9 respectively, $J(M)$ is maximal and is the unique prime ideal in W containing X^m and Y^m .

Let K be an ideal of W containing X^r and Y^r for some $r \in \mathbb{N}$. By Lemma 3.4, if $0 < r < m$ then $K = W$ and if $r > m$ then $X^m \in K$ and $Y^m \in K$. Hence $J(M)$ is the unique prime ideal in W containing X^r and Y^r for any $r \in \mathbb{N}$.

Now suppose that A is α -simple and that no power of α is inner. Let P be a non-zero prime ideal of W . Recall from Definitions 1.2 that $A[Y^{\pm 1}; \alpha]$ and $A[X^{\pm 1}; \alpha^{-1}]$ are the localizations of W at the Ore sets $\{Y^i : i \geq 1\}$ and $\{X^i : i \geq 1\}$ respectively. These rings are simple, by [14, Theorem 1.8.5], so, by [12, Lemma 3.1], there exist r, s such that $X^r \in P$ and $Y^s \in P$. Replacing r and s by their maximum, we can assume that $r = s$. By the above $P = J(M)$. \square

In the case of $U(sl_2)$ and $U_q(sl_2)$, the maximal ideals that arise in the form $J(M)$ are the annihilators of the finite-dimensional simple modules. These are well understood and provide nice illustrations of the theory developed above.

Example 3.11. Let R be as in Example 2.6. Thus $\text{char}(\mathbb{K}) = 0$, $A = \mathbb{K}[t]$, $\alpha(t) = t + 2$, $\rho = 1$, $u = \frac{-1}{4}(t - 1)^2$, $v = t$ and R is the enveloping algebra $U(sl_2)$. We suppose that \mathbb{K} is algebraically closed. Then every height one prime ideal of R has the form $(z - \lambda)R = (xy - (u + \lambda))R$ for some $\lambda \in \mathbb{K}$ and $R/(z - \lambda)R = W(\mathbb{K}[t], \alpha, u + \lambda)$, where $\alpha(t) = t + 2$. For $m \geq 1$, let $M_{m,\lambda} = (u + \lambda)A + \alpha^m(u + \lambda)A$. Note that, for $m \geq 1$, $v^{(m)} = u + \lambda - \alpha^m(u + \lambda) = m(t + m - 1)$ so $v^{(m)}A = (t - (1 - m))A$ and $M_{m,\lambda} = (u + \lambda)A + v^{(m)}A$. Also $u + \lambda \equiv (\lambda - \frac{1}{4}m^2) \pmod{M_{m,\lambda}}$. If $\lambda \neq \frac{1}{4}m^2$ for all $m \in \mathbb{N}$ then $M_{m,\lambda} = A$. On the other hand, if $m \in \mathbb{N}$ and $\lambda = \frac{1}{4}m^2$ then $M_{m,\lambda} = v^{(m)}A = (t + m - 1)A$ is maximal. It follows from Theorems 3.1 and 3.10 that if $y = \frac{1}{4}m^2$ for some $m \geq 1$ then $J(M_{\lambda,m})$ is the unique non-zero prime ideal of $W(\mathbb{K}[t], \alpha, \lambda - \frac{1}{4}(t - 1)^2)$ and that, otherwise, $W(\mathbb{K}[t], \alpha, \lambda - \frac{1}{4}(t - 1)^2)$ is simple.

Example 3.12. Let R be the quantum enveloping algebra $U_q(sl_2)$ as in Example 2.7. Thus $q \in \mathbb{K}^*$ is not a root of unity, $A = \mathbb{K}[t^{\pm 1}]$, $\alpha(t) = q^2t$, $\rho = 1$, and $u = -(q^{-1}t + qt^{-1})/(q - q^{-1})^2$. Every height one prime ideal of R has the form $(z - \lambda)R = (xy - (u + \lambda))R$ for some $\lambda \in \mathbb{K}$ and $R/(z - \lambda)R = W(\mathbb{K}[t^{\pm 1}], \alpha, u + \lambda)$, where $\alpha(t) = q^2t$. For $m \geq 1$,

$$\begin{aligned} v^{(m)} &= u + \lambda - \alpha^m(u + \lambda) \\ &= u((q^{2m-1} - q^{-1})t + (q^{1-2m} - q)t^{-1})/(q - q^{-1})^2 \\ &= \frac{q^{2m-1} - q^{-1}}{(q - q^{-1})^2}(t - q^{2-2m}t^{-1}). \end{aligned}$$

For $m \geq 1$, let $M_{m,\lambda} = (u + \lambda)A + \alpha^m(u + \lambda)A = (u + \lambda)A + v^{(m)}A$. Modulo $v^{(m)}A$, $t^{-1} \equiv q^{2m-2}t$ from which it follows that $M_{m,\lambda}$ contains the ideal $(t^2 - q^{2-2m})A$ and the maximal ideal $(t - \mu)A$ where $\mu = \frac{\lambda(q - q^{-1})^2}{q^{-1} + q^{2m-1}}$. It now follows that

$$M_{m,\lambda} \neq A \Leftrightarrow M_{m,\lambda} \text{ is maximal} \Leftrightarrow \mu^2 = q^{2-2m} \Leftrightarrow \mu = \pm q^{1-m} \Leftrightarrow \lambda = \pm \frac{q^{-m} + q^m}{(q - q^{-1})^2}.$$

If n and m are distinct positive integers then, as $q^{-m} + q^m = q^{-n} + q^n \Rightarrow (q^m - q^n)(1 - q^{-(m+n)}) = 0$ which is impossible as q is not of unity. It now follows from Theorems 3.1 and 3.10 that, for each $m \geq 1$, there are two values of λ for which the exceptional maximal ideal $J(M_{m,\lambda})$ exists. Together with 0 and the ideals $(z - \lambda)R$, these are all the prime ideals of R .

In the next example, the exceptional maximal ideals $J(M)$ have infinite codimension over \mathbb{K} and so are not annihilators of finite-dimensional simple modules.

Example 3.13. Let $p \geq 1$ be odd, let $q \in \mathbb{K}^*$ and suppose that q is not a root of unity. Let $R = R(A, \alpha, v, 1)$ and Ω be as in Example 2.8. We have seen that the height one prime ideals of R are the ideals $(\Omega - \lambda)R$ and, in Corollary 3.3 that $(\Omega - \lambda)R$ is maximal unless $\lambda = \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$ for some $m \in \mathbb{N}$. To complete the analysis of the spectrum of R , let $m \in \mathbb{N}$ and let $\lambda = \pm q^{\frac{p-2m+1}{4}}(q^m + 1)$. It follows from Theorem 3.10 together with Lemma 3.2 and its proof that if then $R/(\Omega - \lambda)R$ has a unique non-zero prime ideal $J((z_p \pm q^{\frac{p+2m-3}{4}})A)$. Therefore the prime spectrum of R consists of 0, the height one prime ideals $(\Omega - \lambda)S$,

$\lambda \in \mathbb{K}$, and countably many height two prime ideals $F_{m,1} = \pi^{-1}(J((z_p - q^{\frac{p+2m-3}{4}})A))$ and $F_{m,-1} = \pi^{-1}(J((z_p + q^{\frac{p+2m-3}{4}})A))$ where $m \in \mathbb{N}$ and each $\pi : S \rightarrow R/(\Omega - \lambda)R$ is the appropriate canonical epimorphism.

4. GOLDIE RANK

In Examples 3.11, 3.12 and 3.13, the height one prime ideals are principal, generated by translates of the Casimir element, all but countably many of these are maximal and the other maximal ideals have height two. For $U(sl_2)$ in 3.11 and $U_q(sl_2)$ in 3.12, the height two maximals are annihilators of finite-dimensional simple modules and so the factor rings are matrix rings over \mathbb{K} . For $U(sl_2)$, there is one simple module of each dimension $d \in \mathbb{N}$ and so there is a unique height two maximal ideal of Goldie rank d . For $U_q(sl_2)$, there are two height two maximal ideals of Goldie rank d . In Example 3.13, the simple factor rings $R/F_{m,1}$ and $R/F_{m,-1}$ are infinite-dimensional and hence not isomorphic to matrix rings over \mathbb{K} . It is the purpose of this section to show that, in the situation of Theorem 3.10, but with the further condition that A/M is a right Ore domain, the quotient $R/J(M)$ has Goldie rank m . So let $W = W(A, \alpha, u)$ be a generalized Weyl algebra, with u central and regular in A , such that, for some fixed $m \in \mathbb{N}$, $M := Au + A\alpha^m(u)$ is such that A/M is a simple right Ore domain and $Au + A\alpha^i(u) = A$ for $i \in \mathbb{N} \setminus \{m\}$.

Notation 4.1. In the above situation, for $0 \leq i \leq m-1$, let $M_i = \alpha^{-i}(M) = A\alpha^{-i}(u) + A\alpha^{m-i}(u) = \alpha^{-i}(u)A + \alpha^{m-i}(u)A$. Thus each M_i is a maximal ideal. As the generators $\alpha^s(u)$ are central, $M_i M_j = M_j M_i$ for $0 \leq i, j \leq m-1$. Also M_0, M_1, \dots, M_{m-1} are distinct for if $0 \leq i < j < m$ and $\alpha^{-i}(M) = M_i = M_j = \alpha^{-j}(M)$ then $\alpha^{j-i}(u) \in \alpha^{j-i}(M) = M$, which is impossible as $Au + A\alpha^{j-i}(u) = A$. So the following result applies.

Lemma 4.2. *Let R be a ring with m commuting distinct maximal ideals M_0, M_1, \dots, M_{m-1} . Let $0 \leq i_1, \dots, i_r, j_1, \dots, j_s, k_1, \dots, k_t < m$ be distinct integers.*

- (i) $M_{i_1} \dots M_{i_r} + M_{j_1} \dots M_{j_s} = R$.
- (ii) $M_{k_1} \dots M_{k_t} M_{i_1} \dots M_{i_r} + M_{k_1} \dots M_{k_t} M_{j_1} \dots M_{j_s} = M_{k_1} \dots M_{k_t}$.
- (iii) $M_0 \cap M_1 \cap \dots \cap M_{m-1} = M_0 M_1 \dots M_{m-1}$.

Proof. (i) Suppose not. Then there exists a maximal ideal M such that $M_{i_1} \dots M_{i_r} + M_{j_1} \dots M_{j_s} \subseteq M$. As M is prime, there exist $1 \leq a \leq r$ and $1 \leq b \leq s$ such that $M_{i_a} \subseteq M$ and $M_{j_b} \subseteq M$. But then, by maximality, $M_{i_a} = M = M_{j_b}$, contrary to the hypotheses.

(ii) This is immediate from (i) and the law $I(J + K) = IJ + IK$.

(iii) We proceed by induction on m . It is certainly true when $m = 1$ so we may assume that $m > 1$ and that $M_1 \dots M_{m-1} = M_1 \cap \dots \cap M_{m-1}$. Let $J = M_1 \cap \dots \cap M_{m-1} = M_1 \dots M_{m-1}$. Then $M_0 + J = R$, by (i), so, as the M_i 's commute,

$$M_0 \cap J = (M_0 \cap J)(M_0 + J) \subseteq JM_0 + M_0J = M_0J \subseteq M_0 \cap J,$$

whence $M_0 \cap M_1 \cap \dots \cap M_{m-1} = M_0 M_1 \dots M_{m-1}$. □

Our aim now is to find m uniform right ideals of the \mathbb{Z} -graded ring $W/J(M)$ whose sum is direct and equal to $W/J(M)$.

Notation 4.3. Let $0 \leq i \leq j \leq m-1$ and let $i \leq r \leq j$. Then we shall denote the product $M_i M_{i+1} \dots M_{r-1} M_{r+1} \dots M_j$ by $\Pi(M, i, \widehat{r}, j)$ and the product $M_i M_{i+1} \dots M_j$ by $\Pi(M, i, j)$.

If $r < i$ or $r > j$ then $\Pi(M, i, \widehat{r}, j)$ should be interpreted as $\Pi(M, i, j)$ and if $i > j$ then $\Pi(M, i, j) = A$.

The components $(W/J(M))_d$ and $(W/J(M))_{-d}$ are 0 if $d \geq m$. If $0 \leq d < m$ then, by Lemma 4.2(iii),

$$(W/J(M))_d = AY^d/\Pi(M, 0, m-1-d)Y^d \text{ and } (W/J(M))_{-d} = AX^d/\Pi(M, d, m-1)X^d.$$

Each $(W/J(M))_d$ is an A - A -bimodule while, in accordance with the proof of Proposition 3.5, right and left multiplication by Y , respectively X , give well-defined maps $(W/J(M))_d \rightarrow (W/J(M))_{d+1}$, respectively $(W/J(M))_d \rightarrow (W/J(M))_{d-1}$.

For $0 \leq r \leq m-1$, let $J^{(r)}$ be the graded right ideal $(\Pi(M, 0, \widehat{r}, m-1)W + J(M))/J(M)$ of $W/J(M)$. The 0-component of $J^{(r)}$ is $\Pi(M, 0, \widehat{r}, m-1)/\Pi(M, 0, m-1)$. If $d > m-r-1$ then $J_d^{(r)} = 0$ and if $1 \leq d \leq m-r-1$ then $J_d^{(r)} = \Pi(M, 0, \widehat{r}, m-1-d)Y^d/\Pi(M, 0, m-1-d)Y^d$. If $r < d$ then $J_{-d}^{(r)} = 0$ and if $1 < d \leq r$, then $J_{-d}^{(r)} = \Pi(M, d, \widehat{r}, m-1)X^d/\Pi(M, d, m-1)X^d$.

Lemma 4.4. *The sum $J^{(0)} + J^{(1)} + \dots + J^{(m-1)}$ is direct and equal to $W/J(M)$.*

Proof. It suffices to show that, for $-m < d < m$, $J_d^{(0)} + J_d^{(1)} + \dots + J_d^{(m-1)} = W/J(M)_d$ and that the sum is direct.

Let $0 < s < m$. Then, by repeated use of Lemma 4.2(ii),

$$\begin{aligned} & J_0^{(0)} + J_0^{(1)} + \dots + J_0^{(s)} \\ &= (\Sigma_{j=0}^s \Pi(M, 0, \widehat{j}, m-1))/\Pi(M, 0, m-1) \\ &= \Pi(M, s+1, m-1)/\Pi(M, 0, m-1). \end{aligned}$$

Similar calculations show that if $0 < d < m$ then

$$\begin{aligned} & J_d^{(0)} + \dots + J_d^{(s)} \\ &= (\Sigma_{j=0}^s \Pi(M, 0, \widehat{j}, m-d-1))Y^d/\Pi(M, 0, m-d-1)Y^d \\ &= \Pi(M, s+1, m-d-1)Y^d/\Pi(M, 0, m-d-1)Y^d \end{aligned}$$

and, with $t = \max\{s+1, d\}$,

$$\begin{aligned} & J_{-d}^{(0)} + \dots + J_{-d}^{(s)} \\ &= (\Sigma_{j=0}^s \Pi(M, d, \widehat{j}, m-1))X^d/\Pi(M, d, m-1)X^d \\ &= \Pi(M, t, m-1)X^d/\Pi(M, d, m-1)X^d. \end{aligned}$$

Taking $s = m-1$ above, $J_0^{(0)} + J_0^{(1)} + \dots + J_0^{(m-1)} = A/\Pi(M, 0, m-1) = (W/J(M))_0$ and, for $0 < d < m$, $J_d^{(0)} + J_d^{(1)} + \dots + J_d^{(m-1)} = AY^d/\Pi(M, 0, m-d-1)Y^d = (W/J(M))_d$ and $J_{-d}^{(0)} + J_{-d}^{(1)} + \dots + J_{-d}^{(m-1)} = AX^d/\Pi(M, d, m-1)X^d = (W/J(M))_{-d}$. It follows that $J^{(0)} + J^{(1)} + \dots + J^{(m-1)} = W/J(M)$.

Also, if $s < m-1$ then $J_0^{(s+1)} = \Pi(M, 0, \widehat{s+1}, m-1)/\Pi(M, 0, m-1)$ and if $0 < d < m$ then $J_d^{(s+1)} = \Pi(M, 0, \widehat{s+1}, m-d-1)Y^d/\Pi(M, 0, m-1)Y^d$ and $J_{-d}^{(s+1)} = \Pi(M, d, \widehat{s+1}, m-1)X^d/\Pi(M, d, m-1)X^d$.

Thus, using Lemma 4.2(iii), $(J_d^{(0)} + J_d^{(1)} + \dots + J_d^{(s)}) \cap J_d^{(s+1)} = 0$ for all d and so $(J^{(0)} + J^{(1)} + \dots + J^{(s)}) \cap J^{(s+1)} = 0$, whence the sum $J^{(0)} + J^{(1)} + \dots + J^{(m-1)}$ is direct. \square

Lemma 4.5. For $0 \leq r \leq m-1$, the right ideal $J^{(r)}$ of W is uniform.

Proof. First consider the A -module $J_0^{(r)}$. Using Lemma 4.2,

$$\begin{aligned} J_0^{(r)} &= \Pi(M, 0, \widehat{r}, m-1) / \Pi(M, 0, m-1) \\ &= \Pi(M, 0, \widehat{r}, m-1) / M_r \cap \Pi(M, 0, \widehat{r}, m-1) \\ &\simeq (M_r + \Pi(M, 0, \widehat{r}, m-1)) / M_r \\ &= A / M_r. \end{aligned}$$

Let $d > 0$ be such that $J_d^{(r)} \neq 0$. Thus $d \leq m-r-1$. Let $h = aY^d + \Pi(M, 0, m-d-1)Y^d \in J_d^{(r)}$, where $a \in \Pi(M, 0, \widehat{r}, m-d-1)$, and suppose that, in $J_{d-1}^{(r)}$, $hX = 0$, that is $a\alpha^d(u)Y^{d-1} + \Pi(M, 0, m-d)Y^{d-1} = 0$. Then $a\alpha^d(u) \in M_r$ so either $a \in M_r$ or $\alpha^d(u) \in M_r$. But $0 < d+r < m$ and $\alpha^i(u) \notin M$ for $0 < i < m$ so $\alpha^d(u) \notin M_r = \alpha^{-r}(M)$. Therefore $a \in M_r$ so $a \in M_r \cap \Pi(M, 0, \widehat{r}, m-d-1) = \Pi(M, 0, m-d-1)$ (by Lemma 4.2(iii)) and $j = 0$. It follows that if $0 \neq h \in J_d^{(r)}$ then $0 \neq hX^d \in J_0^{(r)}$. A similar argument shows that if $0 \neq h \in J_{-d}^{(r)}$ then $0 \neq hY^d \in J_0^{(r)}$. Therefore if $0 \neq j \in J^{(r)}$ then there exists $w \in W$ such that hw has non-zero component in degree 0.

Let $d > 0$ and, as above, let $h = aY^d + \Pi(M, 0, m-d-1)Y^d \in J_d^{(r)}$, where $a \in \Pi(M, 0, \widehat{r}, m-d-1)$. Then $h\alpha^{-(r+d)}(u) = a\alpha^{-r}(u)Y^d + \Pi(M, 0, m-d-1)Y^d = 0$, as $\alpha^{-r}(u) \in M_r$ whence $a\alpha^{-r}(u) \in M_r \cap \Pi(M, 0, \widehat{r}, m-d-1) = \Pi(M, 0, m-d-1)$. Thus $J_d^{(r)}\alpha^{-(r+d)}(u) = 0$. Similarly, $J_{-d}^{(r)}\alpha^{d-r}(u) = 0$. It follows that $J^{(r)}t \subseteq J_0^{(r)}$, where $t = \alpha^{-(r+m-1)}(u) \dots \alpha^{-(r+1)}(u)\alpha^{-(r-1)}(u) \dots \alpha^{-(r-m+1)}(u)$.

Now let $h = a + \Pi(M, 0, m-1) \in J_0^{(r)}$, where $a \in \Pi(M, 0, \widehat{r}, m-1)$. Suppose that $ht = 0$. Then $at \in M_r$ so M_r contains one of $a, \alpha^{-(r+m-1)}(u), \dots, \alpha^{-(r+1)}(u), \alpha^{-(r-1)}(u), \dots, \alpha^{-(r-m+1)}(u)$. But the only integers ℓ such that $\alpha^{-\ell}(u) \in M_r$ are r and $m+r$ so $a \in M_r$ and $h = 0$.

Combining the previous three paragraphs, if $0 \neq j \in J^{(r)}$ then there exists $w \in W$ such that hw has non-zero component in degree 0, hw is homogeneous of degree 0 and $hw \neq 0$.

Finally, let $j_1, j_2 \in J^{(r)} \setminus \{0\}$. By the above, there exist $v_1, v_2 \in W$ such that j_1v_1 and j_2v_2 are non-zero and homogeneous of degree 0. As A/M and A/M_r are isomorphic rings, A/M_r is a right Ore domain and hence $J_0^{(r)}$ is a uniform right A -module. Therefore $j_1W \cap j_2W \neq 0$ and hence $J^{(r)}$ is a uniform right W -module. □

Proposition 4.6. Let $W = W(A, \alpha, u)$ be a generalized Weyl algebra, with u central and regular in A , such that, for some fixed $m \in \mathbb{N}$, $M := Au + A\alpha^m(u)$ is such that A/M is a simple right Ore domain and $Au + A\alpha^i(u) = A$ for $i \in \mathbb{N} \setminus \{m\}$. Then the ring $W/J(M)$ has Goldie rank m .

Proof. This is immediate from Lemmas 4.4 and 4.5. □

Corollary 4.7. Suppose that q is not a root of unity. Let $R = R(A, \alpha, v, 1)$ be as in Examples 2.8 and 3.13. Let $m \in \mathbb{N}$. The prime ideals $F_{m,1}$ and $F_{m,-1}$ of R specified in Example 3.13 have Goldie rank m .

Proof. The conditions of Proposition 4.6 are satisfied by Lemma 3.2 and the fact that A is right Noetherian. □

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