Poisson enveloping algebras and their simplicity

Christopher Fish

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Abstract

We consider the Poisson enveloping algebra $U(A, \Omega(A))$ of a Poisson algebra as defined by Huebschmann, in the case where A is a factor Poisson algebra of a polynomial ring over a field in finitely many variables. We give a presentation for $U(A, \Omega(A))$, and give a condition for $U(A, \Omega(A))$ to be simple in the case where A is a factor ring of a polynomial ring in three variables.

In [Tow13] Towers gives a presentation for the Poisson enveloping algebra $U(A, \Omega(A))$ of a Poisson algebra A defined by Huebschmann in [Hue90] in the case where A is a polynomial algebra. We extend this presentation to give a presentation for $U(A/I, \Omega(A/I))$ where A is a polynomial Poisson algebra and I is a Poisson prime ideal in A, and give a criterion for $U(A/I, \Omega(A/I))$ to be simple when $A = k[x_1, x_2, x_3]$.

The latter question was inspired by the observation that, when A is the semiclassical limit Poisson algebra of the Weyl algebra, i.e. $A = k[x_1, x_2]$ with Poisson bracket $\{x_1, x_2\} = 1$, then $U(A, \Omega(A))$ is isomorphic to the 2^{nd} Weyl algebra (and this extends to higher Weyl algebras also). So (assuming char $k = 0$) A is a simple Poisson algebra and $U(A, \Omega(A))$ is a simple ring. Further, no other Poisson bracket on $k[x_1, x_2]$ results in a simple Poisson algebra, and it is not hard to show (as we will later) that if A is not a simple Poisson algebra then $U(A, \Omega(A))$ is not a simple ring. So it is tempting to conjecture that A is simple if and only if $U(A, \Omega(A))$ is simple. However, as we will show, this turns out not to be the case even when $A = k[x_1, x_2, x_3]$.

At all times k is a fixed algebraically closed field of characteristic 0.

1 Poisson enveloping algebra for quotients of polynomial algebra

Definition 1.1. If A is a commutative algebra the A-module of **Kähler differentials**, $\Omega(A)$, is the free A-module on symbols $\Omega(a)$, all $a \in A$, quotiented by the submodule generated by:

$\Omega(a + b) - \Omega(a) - \Omega(b)$ for $a, b \in A$

$\Omega(\alpha)$ for $\alpha \in k$

$\Omega(ab) - a\Omega(b) - b\Omega(a)$ for $a, b \in A$.

The map $\Omega : A \to \Omega(A)$ is called the **universal derivation** of A.

If A is a Poisson algebra, $\Omega(A)$ can be made into a Lie algebra by defining

$[a\Omega(x), b\Omega(y)] = a\{x, b\}\Omega(y) + b\{a, y\}\Omega(x) + ab\Omega(\{x, y\})$

Lemma 1.2. *[PS10, Remark after 6.3.D] If $A = k[x_1, \ldots, x_n]$ is a polynomial ring and I is a prime ideal in A then*

$$\Omega(A/I) \cong \frac{\bigoplus_{i=1}^{n} \Omega(x_i)A}{\langle \Omega(f_1), \ldots, \Omega(f_r) \rangle}$$

where $\Omega(x_i)$ are symbols, and $\Omega(f) = \Sigma_{i=1}^{n} \frac{\partial f}{\partial x_i} \Omega(x_i)$.

Definition 1.3. ([Tow13, 1.1]). If A is a Poisson algebra whose underlying associative algebra is $k[x_1, \ldots, x_n]$ then we define an associative algebra $P(A)$ generated by symbols $y_1, \ldots, y_n$ and $\Omega(y_1), \ldots \Omega(y_n)$ subject to the relations:

$y_i y_j - y_j y_i = 0$ for $1 \leq i, j \leq n$;

$\Omega(y_i) y_j - y_j \Omega(y_i) = \iota_A(\{x_i, x_j\})$ for $1 \leq i, j \leq n$;

$\Omega(y_i)\Omega(y_j) - \Omega(y_j)\Omega(y_i) = \Sigma_{k=1}^{n} \iota_A \left(\frac{\partial \{x_i, x_j\}}{\partial x_k} \right) \Omega(y_k)$ for $1 \leq i, j \leq n$;

where $\iota_A : A \to P(A)$ is the algebra homomorphism defined by $\iota_A(x_i) = y_i$.

The homomorphism ι_A makes $P(A)$ into an A-module. Let $\iota_{\Omega(A)} : \Omega(A) \to P(A)$ be the module homomorphism defined by $\Omega(x_i) = \Omega(y_i)$.

Lemma 1.4. *If A is a Poisson algebra whose underlying associative algebra is $k[x_1, \ldots, x_n]$ and $I = \langle f_1, \ldots, f_r \rangle$ is a Poisson prime ideal in A then the left ideal $P(I)$ of $P(A)$ generated by $\iota_A(f_1)$, $\ldots$, $\iota_A(f_r)$ and $\iota_{\Omega(A)}(\Omega(f_1))$, $\ldots$, $\iota_{\Omega(A)}(\Omega(f_r))$ is a two-sided ideal of $P(A)$.*

Proof. We note that, for $b \in A$:

$$\Omega(y_i)\iota_A(b) - \iota_A(b)\Omega(y_i) = \iota_A(\{x_i, b\});$$
$$\iota_{\Omega(A)}(\Omega(b))y_i - y_i \iota_{\Omega(A)}(\Omega(b)) = \iota_A(\{b, x_i\});$$
$$\Omega(y_i)\iota_{\Omega(A)}(\Omega(b)) - \iota_{\Omega(A)}(\Omega(b))\Omega(y_i) = \iota_{\Omega(A)}(\Omega(\{x_i, b\})).$$

Where b is a monomial, each of these can be shown using induction on the length of the monomial, and then this can be extended to arbitrary b by linearity.

Applying these with $b = f_i$ and using the fact that I is a Poisson ideal, we get that $\iota_A(f_i)y_j$, $\iota_A(f_i)\Omega(y_j)$, $\iota_{\Omega(A)}(\Omega(f_i))y_j$, and $\iota_{\Omega(A)}(\Omega(f_i))\Omega(y_j)$ are all in $P(I)$ for all $1 \leq i \leq r$, $1 \leq j \leq n$, and thus that $P(I)$ is a two-sided ideal of $P(A)$. □

From now on we will work in the setting of Lemma 1.4.

Definition 1.5. Let $P(A/I) := P(A)/P(I)$.

We define an algebra homomorphism $\iota_{A/I} : A/I \to P(A/I)$ by $\iota_{A/I}(x_i) = y_i$. This is clearly well-defined, and makes $P(A/I)$ into an A/I-module.

We define an A/I-module homomorphism $\iota_{\Omega(A/I)} : \Omega(A/I) \to P(A/I)$ by $\iota_{\Omega(A/I)}(\Omega(x_i)) = \Omega(y_i)$. This is also clearly well-defined.

Remark 1. If I is the zero ideal in A then $P(I)$ is the zero ideal in $P(A)$, and so $P(A/0) = P(A)$, $\iota_{A/0} = \iota_A$, and $\iota_{\Omega(A/0)} = \iota_{\Omega(A)}$.

Lemma 1.6. *If we consider $P(A/I)$ as a Lie algebra with the commutator bracket, $\iota_{\Omega(A/I)}$ is a Lie algebra homomorphism.*

Proof. This is identical to the proof of [Tow13, Lemma 1]. □

Lemma 1.7. *The Poisson enveloping algebra of [Hue90], which he denotes $U(A/I, \Omega(A/I))$, is isomorphic to $P(A/I)$.*

Proof. This is identical to the proof of [Tow13, Lemma 2], except we must additionally check that both $\phi_{A/I}(f_i) = 0$ and $\phi_{\Omega(A/I)}(\Omega(f_i)) = 0$ - but this is immediate since $f_i = 0$ in A/I and $\Omega(f_i) = 0$ in $\Omega(A/I)$. □

2 Simplicity of $P(A/I)$

Definition 2.1. If a Poisson algebra R is such that $\Omega(R)$ is projective as an R-module then we say R is **almost smooth**.

If A is a Poisson algebra whose underlying underlying commutative ring is a polynomial ring in finitely many variables over k, and I is a Poisson ideal in A then we say A/I is a **Poisson algebra of finite type**.

such that A/I is almost smooth then we say A/I is a **almost smooth Poisson algebra of finite type**.

The terminology "almost smooth" comes from the fact that if A/I is smooth in the sense of [Wei95, 9.3.2], then A/I is almost smooth ([Wei95, 9.3.14]). In particular, if $I = 0$ then A/I is smooth, and so almost smooth.

Remark 2. Given a polynomial ring A and an ideal I of A, it is an open question **cite** whether there exist Poisson structures on A/I which do not arise from a Poisson structure on A such that I is a Poisson ideal of A.

Corollary 2.2. *Let A/I be a Poisson algebra of finite type. Then $\iota_{A/I}$ is injective; if A/I is almost smooth then $\iota_{\Omega(A/I)}$ is also injective.*

Proof. This is immediate from [Hue90, Remark after Equation 1.8.3] and [Hue90, Corollary 1.10] given Lemma 1.7. □

We will therefore identify elements of A/I and $\Omega(A/I)$ with their images in $P(A/I)$, using x_i for $y_i = \iota_{A/I}(x_i)$ and $\Omega(x_i)$ for $\Omega(y_i) = \iota_{\Omega(A/I)}(\Omega(x_i))$. The defining relations for $P(A/I)$ are then:

$$x_i x_j - x_j x_i = 0 \quad \text{for } 1 \leq i, j \leq n;$$
$$\Omega(x_i)x_j - x_j\Omega(x_i) = \{x_i, x_j\} \quad \text{for } 1 \leq i, j \leq n;$$
$$\Omega(x_i)\Omega(x_j) - \Omega(x_j)\Omega(x_i) = \Sigma_{k=1}^{n} \frac{\partial\{x_i, x_j\}}{\partial x_k}\Omega(x_k) \quad \text{for } 1 \leq i, j \leq n;$$

Lemma 2.3. *Let A/I be a Poisson algebra of finite type.*

(i) Let J be an ideal in $P(A/I)$. Then $J \cap A/I$ is a Poisson ideal in A/I.

(ii) Let J be a Poisson ideal in A/I. Then the left ideal $P(A/I)J$ is a two-sided ideal in $P(A/I)$.

Proof. These both follow from $\Omega(x_i)b - b\Omega(x_i) = \{x_i, b\}$:

(i) Certainly $J \cap A/I$ is an ideal of A/I; if $b \in J \cap A/I$ then $\{x_i, b\} = \Omega(x_i)b - b\Omega(x_i) \in J$, so $\{x_i, b\} \in J \cap A/I$; therefore $J \cap A/I$ is a Poisson ideal of A/I.

(ii) If $b \in J$, then $b\Omega(x_i) = \Omega(x_i)b - \{x_i, b\} \in P(A/I)J$; and $bx_i = x_ib \in P(A/I)J$. Thus $P(A/I)J$ is a right ideal of $P(A/I)$, and so a two-sided ideal. □

As in [Hue90, Remark before Theorem 1.9] and [Rin63, Remark before Theorem 3.1], we can filter $P(A/I)$ with $P_{-1}(A/I) = 0$, and for $n \geq 0$, $P_n(A/I)$ being the left A/I-submodule of $P(A/I)$ generated by products of at most n elements from $\Omega(A/I)$. We note that since, for $a \in A/I$ and $z \in P_n(A/I)$ we have $az - za \in P_{n-1}(A/I)$, the left and right A/I-module structures on the associated graded ring $gr\ P(A/I)$ coincide, and so it has the structure of a commutative graded A/I-algebra.

Lemma 2.4. *Let A/I be an almost smooth Poisson algebra of finite type such that I is Poisson prime. Then every non-zero element of A/I is a regular element of $P(A/I)$.*

Proof. Since by assumption $\Omega(A/I)$ is projective as an A/I-module, by [Hue90, Theorem 1.9], $gr\ P(A/I) \cong S_{A/I}[\Omega(A/I)]$, where the latter denotes the symmetric A/I-algebra on $\Omega(A/I)$. Also, since $\Omega(A/I)$ is projective as an A/I-module, $S_{A/I}[\Omega(A/I)]$ is also projective, and thus torsion-free. Thus gr $P(A/I)$ is torsion-free as a left or right A/I-module, and so $P(A/I)$ is also torsion-free as a left or right A/I-module, so a is a regular element of $P(A/I)$ for all non-zero $a \in A/I$. □

Lemma 2.5. *Assume I is Poisson prime and A/I is a regular ring. Let $\mathcal{X} = \{f^a : a \in \mathbb{N}\}$ for some $0 \neq f \in A/I$. Then $\mathcal{X}$ is an Ore set in $P(A/I)$.*

Proof. We note that

$f^{a+1}\Omega(x_i) - \Omega(x_i)f^{a+1} = (a+1)f^a\{f, x_i\}$

and so

$f^a(f\Omega(x_i) - (a+1)\{f, x_i\}) = \Omega(x_i)f^{a+1}$.

Therefore $xP(A/I) \cap r\mathcal{X} \neq \emptyset$ holds when $r = \Omega(x_i)$ (for any $1 \leq i \leq r$), for any $x \in \mathcal{X}$.

Using the fact that $\mathcal{X} = \{f^a : a \in \mathbb{N}\}$, this can be extended to monomials in the $\Omega(x_i)$ by induction on the length of the monomial, and then to A/I-linear sums of such monomials, i.e. to all of $P(A/I)$. □

In the above situation we will write $P(A/I)_f := P(A/I)_{\mathcal{X}}$.

Lemma 2.6. *Let $I = \langle u \rangle$, $u \neq 0$, be a principal Poisson ideal in $A = k[x_1, x_2, x_3]$, such that A/I is Poisson simple. Then there exists $\pi \in S_3$ such that $\{x_{\pi(1)}, x_{\pi(2)}\} \neq 0$ (in A/I), and $\frac{\partial u}{\partial x_{\pi(3)}} \neq 0$ (in A/I).*

Proof. Suppose not. Since I is principal in A it cannot be maximal, and so since A/I is Poisson simple, the Poisson bracket on A/I must be non-zero. Wlog, $\{x_1, x_2\} \neq 0$. So $\frac{\partial u}{\partial x_3} = 0$. $u \neq 0$, so wlog, $\frac{\partial u}{\partial x_1} \neq 0$. So $\{x_2, x_3\} = 0$. If $\frac{\partial u}{\partial x_2} \neq 0$ also, then $\{x_1, x_3\} = 0$, and so x_3 is Poisson central in A/I, and non-zero since $\frac{\partial u}{\partial x_3} = 0$ - and so A/I cannot be Poisson simple. So we must have $\frac{\partial u}{\partial x_2} = 0$, i.e. $u \in k[x_1]$. Since I is prime and k is algebraically closed, we must have $u = x_1 - \lambda$, some $\lambda \in k$. But then A/I is residually null, and so cannot be Poisson simple. So we have a contradiction. □

Lemma 2.7. *Let $I = \langle u \rangle$, $u \neq 0$, be a principal Poisson ideal in $A = k[x_1, x_2, x_3]$, such that A/I is Poisson simple and almost smooth. Then $P(A/I)$ is simple.*

Proof. From the previous lemma, we can assume without loss of generality that $\frac{\partial u}{\partial x_3} \neq 0$ and $\{x_1, x_2\} \neq 0$. Let $R = P(A/I)_{\frac{\partial u}{\partial x_3}}$ and $S = (A/I)_{\frac{\partial u}{\partial x_3}}$. (These localisations exists by Lemmas 2.4 and 2.5). Extend the Poisson bracket on A/I to S using the quotient rule for derivations, and identify S as a subring of R in the natural way.

Define a derivation δ of S by $\delta(s) = \{x_2, s\}$. Define an automorphism α of $S[z_2; \delta]$ by $\alpha(s) = s$ for $s \in S$, $\alpha(z_2) = z_2 + \frac{\partial\{x_1,x_2\}}{\partial x_1} + \frac{\partial\{x_1,x_2\}}{\partial x_3}(\frac{\partial u}{\partial x_3})^{-1}\frac{\partial u}{\partial x_1}$. Define an α-derivation γ of $S[z_2; \delta]$ by $\gamma(s) = \{x_1, s\}$ for $s \in S$, $\gamma(z_2) = \left(\frac{\partial\{x_1,x_2\}}{\partial x_2} + \frac{\partial\{x_1,x_2\}}{\partial x_3}\left(\frac{\partial u}{\partial x_3}\right)^{-1}\frac{\partial u}{\partial x_2}\right) z_2$. Let $T = S[z_2; \delta][z_1; \alpha, \gamma]$.

Finally, define a ring homomorphism $\eta : T \to R$ by $\eta(s) = s$ for $s \in S$, $\eta(z_1) = \Omega(x_1)$, $\eta(z_2) = \Omega(x_2)$. Using that $\frac{\partial u}{\partial x_1}\Omega(x_1) + \frac{\partial u}{\partial x_2}\Omega(x_2) + \frac{\partial u}{\partial x_3}\Omega(x_3) = 0$ in R, we see that η is well-defined and surjective; also, η is injective on S.

Let P be a prime ideal in T. Let $t = p_r z_1^r + ... + p_0$, where $p_i \in S[z_2; \delta]$, be an element of P of minimal degree r in z_1. But x_2 commutes with everything in $S[z_2; \delta]$, and so $[t, x_2] = rp_r\{x_1, x_2\}z_1^{r-1}+$ terms of smaller degree. By minimality of r and the fact that $\{x_1, x_2\} \neq 0$, we must have $r = 0$. Similarly, by considering an element of P of minimal degree in z_2, P must contain a non-zero element of S. But then any prime ideal of R must also contain a non-zero element of S. But since A/I is Poisson simple, any such ideal of R is all of R. Thus R is simple, and so $P(A/I)$ must be simple also, since any prime ideal of $P(A/I)$ must contain a power of $\frac{\partial u}{\partial x_3}$, but again, since A/I is Poisson simple, any such ideal of $P(A/I)$ is all of $P(A/I)$. □

Theorem 2.8. *Let $A = k[x_1, x_2, x_3]$, and let I be a Poisson prime ideal in A such that A/I is almost smooth. Then $P(A/I)$ is simple if and only if A/I is Poisson simple and $I \neq 0$.*

Proof. By Lemma 2.3, if $P(A/I)$ is simple then A/I is Poisson simple. So assume A/I is Poisson simple.

Case 1: height$(I) = 3$. In this case I is maximal and so $A/I \cong k$, $P(A/I) \cong k$, and so $P(A/I)$ is simple.

Case 2: height$(I) = 2$. This cannot happen, as in this case A/I is residually null - i.e. has zero Poisson bracket - by [JO12, Proposition 3.2], and so cannot be Poisson simple.

Case 3: height$(I) = 1$. In this case I is principal, since A is a UFD, say $I = \langle u \rangle$. Therefore $P(A/I)$ is simple by Lemma 2.7.

Case 4: height$(I) = 0$, so $I = 0$.

In this case $P(A/I)$ is not simple: let $\Lambda = \{x_1, x_2\}\Omega(x_3)+\{x_2, x_3\}\Omega(x_1)+\{x_3, x_1\}\Omega(x_2))$. Then we claim Λ is normal in $P(A/I)$. We use the notation $[a, b] = ab-ba$ for $a, b \in P(A/I)$.

Firstly, $[\Lambda, x_3] = \{x_2, y_3\}\{x_1, x_3\} + \{x_3, x_1\}\{x_2, x_3\} = 0$, and similiarly $[\Lambda, x_1] = [\Lambda, x_2] = 0$. Then:

$$
\begin{aligned}
&[\Lambda, \Omega(x_3)] - \left(\frac{\partial\{x_2,x_3\}}{\partial x_2} - \frac{\partial\{x_3,x_1\}}{\partial x_1}\right)\Lambda \\
&= \{\{x_1,x_2\},x_3\}\Omega(x_3) \\
&\quad + \frac{\partial\{x_2,x_3\}}{\partial x_1}\{x_1,x_3\}\Omega(x_1) + \frac{\partial\{x_2,x_3\}}{\partial x_2}\{x_2,y_3\}\Omega(y_1) \\
&\quad + \{x_2,x_3\}\left(\frac{\partial\{x_1,x_3\}}{\partial x_1}\Omega(x_1) + \frac{\partial\{x_1,x_3\}}{\partial x_2}\Omega(x_2) + \frac{\partial\{x_1,x_3\}}{\partial x_3}\Omega(x_3)\right) \\
&\quad + \frac{\partial\{x_3,x_1\}}{\partial x_1}\{x_1,x_3\}\Omega(x_2) + \frac{\partial\{x_3,x_1\}}{\partial x_2}\{x_2,y_3\}\Omega(x_2) \\
&\quad + \{x_3,x_1\}\left(\frac{\partial\{x_2,x_3\}}{\partial x_1}\Omega(x_1) + \frac{\partial\{x_2,x_3\}}{\partial x_2}\Omega(x_2) + \frac{\partial\{x_2,x_3\}}{\partial x_3}\Omega(x_3)\right) \\
&\quad - \left(\frac{\partial\{x_2,x_3\}}{\partial x_2} - \frac{\partial\{x_3,x_1\}}{\partial x_1}\right)\{x_2,x_3\}\Omega(x_1) \\
&\quad - \left(\frac{\partial\{x_2,x_3\}}{\partial x_2} - \frac{\partial\{x_3,x_1\}}{\partial x_1}\right)\{x_3,x_1\}\Omega(x_2) \\
&\quad - \left(\frac{\partial\{x_2,x_3\}}{\partial x_2} - \frac{\partial\{x_3,x_1\}}{\partial x_1}\right)\{x_1,x_2\}\Omega(x_3) \\
&= \Bigg(\{\{x_1,x_2\},x_3\} + \{x_2,x_3\}\frac{\partial\{x_1,x_3\}}{\partial x_3} + \{x_3,x_1\}\frac{\partial\{x_2,x_3\}}{\partial x_3} \\
&\quad - \{x_1,x_2\}\frac{\partial\{x_2,x_3\}}{\partial x_2} + \{x_1,x_2\}\frac{\partial\{x_3,x_1\}}{\partial x_1}\Bigg)\Omega(x_3) \\
&= \Bigg(\{\{x_1,x_2\},x_3\} + \{x_1,x_2\}\frac{\partial\{x_3,x_1\}}{\partial x_1} + \{x_3,x_2\}\frac{\partial\{x_3,x_1\}}{\partial x_3} \\
&\quad + \{x_2,x_1\}\frac{\partial\{x_2,x_3\}}{\partial x_2} + \{x_3,x_1\}\frac{\partial\{x_2,x_3\}}{\partial x_3}\Bigg)\Omega(x_3) \\
&= (\{\{x_1,x_2\},x_3\} + \{\{x_3,x_1\},x_2\} + \{\{x_2,x_3\},x_1\})\Omega(x_3) \\
&= 0 \qquad\qquad \text{by Jacobi.}
\end{aligned}
$$

Similiarly, $[\Lambda, \Omega(x_1)] = \left(\frac{\partial\{x_3,x_1\}}{\partial x_3} - \frac{\partial\{x_1,x_2\}}{\partial x_2}\right)\Lambda$ and $[\Lambda, \Omega(x_2)] = \left(\frac{\partial\{x_1,x_2\}}{\partial x_1} - \frac{\partial\{x_2,x_3\}}{\partial x_3}\right)\Lambda$.

Therefore Λ is normal in $P(A/I)$; by considering degree it is neither a unit nor zero, and so it generates a non-trivial two-sided ideal in $P(A/I)$. □

Example 1. Take the Poisson bracket on $A = k[x_1, x_2, x_3]$ defined by $\{x_1, x_2\} = 0$, $\{x_i, x_3\} = \delta(x_i)$, where δ is a simple derivation of $k[x_1, x_2]$. In this case A is a simple Poisson algebra, so $P(A)$ is not simple - the element $\Lambda = \iota_A(\delta(x_2))\Omega(y_1) - \iota_A(\delta(x_1))\Omega(y_2)$ is normal.

Example 2. The following is known as a **qm-exact bracket** on $\mathbb{C}[x_1, x_2, x_3]$. These were studied, including a complete description of the Poisson prime ideal structure, in [JO12].

Let $A = \mathbb{C}[x_1, x_2, x_3]$. Let $s, t \in A \setminus \{0\}$ be coprime. Then $\{x, y\} = ts_z - st_z$, $\{y, z\} = ts_x - st_x$, $\{z, x\} = ts_y - st_y$ defines a Poisson bracket on A. The height one prime ideals in A are the ideals uA, where u is an irreducible component of $\lambda s - \mu t$ for some $(\lambda, \mu) \in \mathbb{CP}^1$.

If u has any singular points then those singular points correspond to maximal Poisson ideals of A, and so if A/uA is Poisson simple then it must also be regular. So $P(A/uA)$ is simple whenever A/uA is Poisson simple.

Example 3. Taking $s = x_3$, $t = 1$, $(\lambda, \mu) = (1, 0)$ makes A/uA the semiclassical limit of the first Weyl algebra, which is simple (and regular); in this case $P(A/I)$ turns out to be the second Weyl algebra.

References

[Hue90] Johannes Huebschmann. Poisson cohomology and quantization. *J. reine angew. Math*, 408:57–113, 1990.

[JO12] David A. Jordan and Sei-Qwon Oh. Poisson brackets and Poisson spectra in polynomial algebras. In *New Trends in Noncommutative Algebra*, Contemporary mathematics. American Mathematical Society, 2012.

[PS10] D.P. Patil and U. Storch. *Introduction to Algebraic Geometry and Commutative Algebra*. IISc lecture notes series. World Scientific, 2010.

[Rin63] George S. Rinehart. Differential forms on general commutative algebras. *Transactions of the American Mathematical Society*, 108(2):pp. 195–222, 1963.

[Tow13] Matthew Towers. Poisson and hochschild cohomology and the semiclassical limit. 2013.

[Wei95] C.A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.