CONNECTED QUANTIZED WEYL ALGEBRAS AND QUANTUM CLUSTER ALGEBRAS

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ABSTRACT. For an algebraically closed field $\mathbb{K}$, we investigate a class of noncommutative $\mathbb{K}$-algebras called *connected quantized Weyl algebras*. Such an algebra has a PBW basis for a set of generators $\{x_1, \ldots, x_n\}$ such that each pair satisfies a relation of the form $x_i x_j = q_{ij} x_j x_i + r_{ij}$, where $q_{ij} \in \mathbb{K}^*$ and $r_{ij} \in \mathbb{K}$, with, in some sense, sufficiently many pairs for which $r_{ij} \neq 0$. For such an algebra it turns out that there is a single parameter q such that each $q_{ij} = q^{\pm 1}$. Assuming that $q \neq \pm 1$, we classify connected quantized Weyl algebras, showing that there are two types *linear* and *cyclic*. When q is not a root of unity we determine the prime spectra for each type. The linear case is the easier, although the result depends on the parity of n, and all prime ideals are completely prime. In the cyclic case, which can only occur if n is odd, there are prime ideals for which the factors have arbitrarily large Goldie rank.

We apply connected quantized Weyl algebras to obtain presentations of two classes of quantum cluster algebras. Let $n \geq 3$ be an odd integer. We present the quantum cluster algebra of a Dynkin quiver of type A_{n-1} as a factor of a linear connected quantized Weyl algebra by an ideal generated by a central element. We also consider the quiver $P_{n+1}^{(1)}$ identified by Fordy and Marsh in their analysis of periodic quiver mutation. When n is odd, we show that the quantum cluster algebra of this quiver is generated by a cyclic connected quantized Weyl algebra in n variables and one further generator. We also present it as the factor of an iterated skew polynomial algebra in $n+2$ variables by an ideal generated by a central element. For both classes, the quantum cluster algebras are simple noetherian.

We establish Poisson analogues of the results on prime ideals and quantum cluster algebras. We determine the Poisson prime spectra for the semiclassical limits of the linear and cyclic connected quantized Weyl algebras and show that, when n is odd, the cluster algebras of A_{n-1} and $P_{n+1}^{(1)}$ are simple Poisson algebras that can each be presented as a Poisson factor of a polynomial algebra, with an appropriate Poisson bracket, by a principal ideal generated by a Poisson central element.

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1. Introduction

This paper is mostly in the context of noncommutative ring theory, in particular skew polynomial rings, classification of prime ideals and applications to quantum cluster algebras. The original motivation can be traced back to the classification of mutation periodic quivers by Fordy and Marsh [11] and to a Poisson algebra P introduced by Fordy [12] in a further study of some such quivers.

The Poisson algebra P is a polynomial algebra in an odd number of indeterminates $x_1, \dots, x_n$ and it may be helpful to think of these arranged cyclically, so that x_1 is adjacent to x_n as well as to x_2. Up to a factor of 2, the Poisson bracket is such that

$$\{x_i, x_{i+1}\} = x_i x_{i+1} - 1, \quad 1 \le i \le n, \tag{1}$$

where x_{n+1} should be interpreted as x_1, and, for $1 \le i, j \le n$ with $j > i+1$,

$$\{x_i, x_j\} = \begin{cases} x_i x_j & \text{if } j-i \text{ is odd,} \\ -x_i x_j & \text{if } j-i \text{ is even.} \end{cases} \tag{2}$$

In the sense of [4, Chapter III.5], this algebra is quantized by the algebra C_n^q generated by $x_1, \dots, x_n$ subject to the relations

$$x_i x_{i+1} - q x_{i+1} x_i = 1 - q, \quad 1 \le i \le n, \tag{3}$$

where x_{n+1} should again be interpreted as x_1, and, for $1 \le i, j \le n$ with $j > i+1$,

$$x_i x_j = \begin{cases} q x_j x_i & \text{if } j-i \text{ is odd,} \\ q^{-1} x_j x_i & \text{if } j-i \text{ is even.} \end{cases} \tag{4}$$

We shall interpret the relation (3) as the defining relation for the quantized Weyl algebra A_1^q generated by x_i and x_{i+1}. This is more commonly written

$$x_i x_{i+1} - q x_{i+1} x_i = 1, \quad 1 \le i \le n, \tag{5}$$

but, unless $q = 1$, the two are isomorphic (by a change of variable) and (5) is less satisfactory both from the point of view of quantization, as it gives a noncommutative algebra on setting $q = 1$, and from the point of view of symmetry, as it is equivalent not to $x_{i+1}x_i - q^{-1}x_i x_{i+1} = 1$ but to $x_{i+1}x_i - q^{-1}x_i x_{i+1} = -q^{-1}$.

It is possible to construct C_n^q as an iterated skew polynomial algebra in $x_1, x_2, \dots, x_n$ over the base field $\mathbb{K}$. If $1 \le m < n$ the intermediate iterated skew polynomial algebra in $x_1, x_2, \dots, x_m$ will be denoted L_m^q. When n is odd, L_n^q and C_n^q both exist and are skew polynomial rings over L_{n-1}^q. As iterated skew polynomial algebras over the base field, L_n^q and, if n is odd, C_n^q are noetherian domains with PBW-bases.

Motivated by the algebras L_n^q and C_n^q, we define a *c*onnected quantized Weyl algebra to be a $\mathbb{K}$-algebra with a finite set $\{x_1, \dots, x_n\}$ of generators such that

- each pair of generators satisfies a relation of the form

  $$x_i x_j = q_{ij} x_j x_i + r_{ij},$$

  where $q_{ij} \in \mathbb{K}^*$ and $r_{ij} \in \mathbb{K}$,

- the standard monomials $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ form a PBW basis,
- the graph G with vertices $x_1, x_2, \dots, x_n$, in which there is an edge between x_i and x_j if and only if $r_{ij} \neq 0$, is connected.

In Section 2 we shall see that, provided at least one $q_{ij} \neq \pm 1$, L_n^q and C_n^q are the only connected quantized Weyl algebras.

In Sections 3 and 4, using a deleting derivations algorithm similar to that applied to quantum matrices by Cauchon [5], we determine, when q is not a root of unity, the prime spectra of L_n^q and, when n is odd, C_n^q. In the case of L_n^q, the hypotheses of [16, Theorem 2.3] are satisfied so all prime ideals are completely prime but we shall see that C_n^q does have prime ideals that are not completely prime. In L_n^q there is a sequence of elements $z_1, \dots, z_n$, defined by the formula $z_j = z_{j-1}x_j - z_{j-2}$, where $z_0 = 1$ and $z_{-1} = 0$, such that z_n is central if n is odd and normal, but not central, if n is even. In the odd case, the non-zero prime ideals of L_n^q are the ideals of the form $(z_n - \lambda)L_n^q$, $\lambda \in \mathbb{K}$, and in the even case they are $z_nL_n^q$ and the ideals of the form $z_nL_n^q + (z_{n-1} - \lambda)L_n^q$, $0 \neq \lambda \in \mathbb{K}$. There is a similar sequence in C_n^q but with z_n replaced by a central element Ω. The prime ideals of C_n^q are the ideals of the form $(\Omega - \lambda)C_n^q$, $\lambda \in \mathbb{K}$, and, for each positive integer m, two prime ideals $F_{m,1}$ and $F_{m,-1}$, such that the factors $C_n^q/F_{m,\pm 1}$ have Goldie rank m. Thus the prime spectrum of L_n^q is akin to those of $U(sl_2)$ and $U_q(sl_2)$ but, unless $n = 3$, the exceptional maximal ideals are not annihilators of finite-dimensional simple modules.

Section 5 determines the $\mathbb{K}$-automorphism groups of L_n^q and C_n^q when $q \neq \pm 1$. Whereas $\mathrm{Aut}_{\mathbb{K}}(L_n^q)$ is isomorphic to the multiplicative group $\mathbb{K}^*$, with each $\lambda \in \mathbb{K}^*$ corresponding to an automorphism with $x_i \mapsto \lambda^{(-1)^i}x_i$, $\mathrm{Aut}_{\mathbb{K}}(C_n^q)$ is cyclic of order $2n$ generated by the product of the $\mathbb{K}$-automorphism of order n such that each $x_i \mapsto x_{i+1}$, where $x_{n+1} = x_1$, and the automorphism of order 2 such that each $x_i \mapsto -x_i$.

In Section 6 we apply connected quantized Weyl algebras to quantum cluster algebras. Useful references for such algebras include [2, 19, 20, 26]. Although there are several papers, for example [19, 20], showing that given noncommutative algebras have quantum cluster algebra structures, there are not many in which a quantum cluster algebra is determined given a particular quiver. For two classes of quivers, we relate the quantum cluster algebra to connected quantized Weyl algebras and obtain presentations in terms of generators and relations. For an even positive integer m we present the quantum cluster algebra for a Dynkin quiver of type A_m as a factor of the linear connected quantized Weyl algebra L_{m+1}^q. For an odd integer $n \geq 3$ we present the quantum cluster algebra for the periodic quiver denoted $P_{n+1}^{(1)}$ in [11] as an extension of the cyclic connected quantized Weyl algebra $C_n^{q^2}$, requiring one further generator, and also as a factor, by the principal ideal generated by a central element, of an iterated skew polynomial ring in $n+2$ variables over the base field $\mathbb{K}$.

Sections 7 and 8 present Poisson analogues of earlier results. In Section 7 the Poisson prime spectra of the semiclassical limits of L_n^q and C_n^q are determined and in Section 8 the cluster algebras of A_{n-1} and $P_{n+1}^{(1)}$ are presented as factors, by Poisson ideals generated by a Poisson central element, of polynomial algebras with appropriate Poisson brackets.

2. Connected quantized Weyl algebras

Throughout $\mathbb{K}$ will denote an algebraically closed field and $q \in \mathbb{K}^*$. We make a fixed choice of one of the square roots of q in $\mathbb{K}$ and denote it by $q^{\frac{1}{2}}$.

Notation 2.1. The (co-ordinate ring of the) quantum plane, R_q, is the $\mathbb{K}$-algebra generated by x and y subject to the relation $xy = qyx$. There is symmetry, up to the transposition of q and q^{-1}, as the relation can be rewritten $yx = q^{-1}xy$. It is well-known that R_q is the skew polynomial ring $\mathbb{K}[y][x; \alpha_q]$ where α_q is the $\mathbb{K}$-automorphism of $\mathbb{K}[y]$ such that $\alpha_q(y) = qy$. As such, it has a PBW basis $\{y^i x^j : i, j \geq 0\}$. By symmetry, $R_q = \mathbb{K}[x][y; \beta_{q^{-1}}]$ where $\beta_{q^{-1}}$ is the $\mathbb{K}$-automorphism of $\mathbb{K}[x]$ such that $\beta_{q^{-1}}(x) = q^{-1}x$ and has a PBW basis $\{x^i y^j : i, j \geq 0\}$.

By the first quantized Weyl algebra $A_1^q(\mathbb{K})$, we mean the $\mathbb{K}$-algebra generated by x and y subject to the relation $xy - qyx = 1 - q$. With α_q and $\beta_{q^{-1}}$ as in the above discussion of the quantum plane, $A_1^q(\mathbb{K})$ has presentations as the skew polynomial ring $\mathbb{K}[y][x; \alpha_q, \delta]$, where $\delta(y) = 1 - q$, and as the skew polynomial ring $\mathbb{K}[x][y; \beta_{q^{-1}}, \delta']$, where $\delta'(y) = 1 - q^{-1}$. As such, it has PBW bases $\{y^i x^j : i, j \geq 0\}$ and $\{x^i y^j : i, j \geq 0\}$.

Remark 2.2. The reader may be more familiar with the relation $xy - qyx = 1$ here. Unless $q = 1$, the two give isomorphic algebras. Indeed, if $r \in \mathbb{K}$, and $xy - qyx = 1$ and $x' = rx$ then $x'y - qyx' = r$ so, by changing generators, the right hand side of the relation $xy - qyx = 1$ can be replaced by any non-zero scalar r. There are two advantages in choosing $1 - q$ in this role. The first is that the relation $xy - qyx = 1 - q$ can be rewritten $yx - q^{-1}xy = 1 - q^{-1}$, giving symmetry, up to the transposition of q and q^{-1}. The second is that setting $q = 1$ yields the commutative algebra $\mathbb{K}[x, y]$, giving rise to a Poisson bracket on $\mathbb{K}[x, y]$ with $\{x, y\} = xy - 1$. This Poisson bracket arises from the quantization procedure outlined in [4, III.5.4] with $A = \mathbb{K}[y, Q^{\pm 1}][x; \alpha, \rho]$ and $h = Q - 1$, where α is the $\mathbb{K}$-automorphism of $\mathbb{K}[y, Q^{\pm 1}]$ such that $\alpha(y) = Qy$ and $\alpha(Q) = Q$, and ρ is the α-derivation of $\mathbb{K}[y]$ such that $\rho(y) = 1 - Q$ and $\delta(Q) = 0$, whence Q and h are central, $xy - Qyx = (1 - Q)$, $xy - yx = h(xy - 1)$, $A/hA \simeq \mathbb{K}[x, y]$ and, if $q \neq 1$, $A/(Q - q)A \simeq A_1^q(\mathbb{K})$.

Loosely speaking, a connected quantized Weyl algebra is a finitely generated $\mathbb{K}$-algebra with a PBW basis in which any two generators satisfy a quantum plane relation or a quantized Weyl relation and there are sufficiently many of the latter. The formal definition is as follows.

Definition 2.3. By a *connected quantized Weyl algebra* over $\mathbb{K}$, we shall mean a $\mathbb{K}$-algebra with generators $x_1, x_2, \ldots, x_n$, $n \geq 2$, satisfying the following properties:

(i) there are scalars $q_{ij} \in \mathbb{K}^*$ and $r_{ij} \in \mathbb{K}$, $1 \leq i \neq j \leq n$, with each $q_{ji} = q_{ij}^{-1}$ and each $r_{ji} = -q_{ij}^{-1} r_{ij}$, such that the relation
$$x_i x_j = q_{ij} x_j x_i + r_{ij}$$
holds;

(ii) the standard monomials $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ form a PBW basis for R;

(iii) the graph with vertices $x_1, x_2, \ldots, x_n$, in which there is an edge between x_i and x_j if and only if $r_{ij} \neq 0$, is connected.

Remark 2.4. The conditions on q_{ji} and r_{ji} ensure that the relations $x_ix_j = q_{ij}x_jx_i + r_{ij}$ and $x_jx_i = q_{ji}x_ix_j + r_{ji}$ are equivalent. The relations $x_ix_j = q_{ij}x_jx_i + r_{ij}$ ensure that the standard monomials $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ span R. The PBW condition therefore ensures that the relations $x_ix_j = q_{ij}x_jx_i + r_{ij}$ form a complete set of defining relations for R.

When $n = 2$, up to isomorphism, the quantized Weyl algebra A_i^q is the only connected quantized Weyl algebra over $\mathbb{K}$. We can take $q_{12} = q$ and $r_{12} = 1 - q$ or, if $q = 1$, $r_{12} = 1$.

Remark 2.5. We can orientate and label the graph in the definition to carry more information. As shown in Figure 1, we orientate an edge representing a relation $x_ix_j = q_{ij}x_jx_i + r_{ij}$ with $r_{ij} \neq 0$ from x_i to x_j with label q_{ij} (or from x_j to x_i with label q_{ji}, but not both).

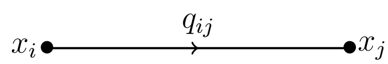

FIGURE 1. orientation of edges

Example 2.6. Let $n \geqslant 1$ and let $q \in \mathbb{K}^*\backslash\{1\}$. Let L_n^q denote the $\mathbb{K}$-algebra generated by $x_1, x_2, \dots, x_n$ subject to the relations

$$x_ix_{i+1} - qx_{i+1}x_i = 1 - q, \quad 1 \leqslant i \leqslant n-1, \tag{6}$$

$$x_ix_j - qx_jx_i = 0, \quad i \geqslant 1,\ i+1 < j \leqslant n,\ j-i \text{ odd}, \tag{7}$$

$$x_ix_j - q^{-1}x_jx_i = 0, \quad i \geqslant 1,\ i+1 < j \leqslant n,\ j-i \text{ even}. \tag{8}$$

In particular $L_1^q = \mathbb{K}[x_1]$ and $L_2^q = A_1^q$. Using [9, Proposition 1], one can show, inductively, that, for $n \geq 2$, L_n^q is the skew polynomial ring $L_{n-1}^q[x_n; \tau_n, \delta_n]$ for a $\mathbb{K}$-automorphism τ_n and a τ_n-derivation δ_n of L_{n-1}^q such that

$$\begin{aligned} \tau_n(x_j) &= q^{(-1)^{n-j}}x_j, && 1 \leqslant j \leqslant n-1, \\ \delta_n(x_j) &= 0, && 1 \leqslant j \leqslant n-2, \\ \delta_n(x_{n-1}) &= 1 - q^{-1}. \end{aligned}$$

Informally, it suffices to show that τ_n and δ_n respect the defining relations of L_{n-1}^q. More formally, one can write L_{n-1}^q as a factor F/I of the free algebra over $\mathbb{K}$ on $n-1$ generators and show that τ_n and δ_n are induced by an appropriate automorphism Γ_n and Γ_n-derivation Δ_n of F such that $\Gamma_n(I) = I$ and $\Delta_n(I) \subseteq I$.

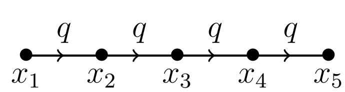

FIGURE 2. L_5^q

As L_n^q is an iterated skew polynomial ring over $\mathbb{K}$, it has a PBW-basis $\{x_1^{a_1}x_2^{a_2}\dots x_n^{a_n} : (a_1, a_2, \dots, a_n) \in \mathbb{N}_0^n\}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Provided $n \geq 2$, L_n^q clearly satisfies the other conditions for a connected quantized Weyl algebra, the relevant graph for the presentation in (6)-(8) being the path graph P_n, and we may refer to it as the *linear* connected quantized Weyl algebra L_n^q. Figure 2 shows the graph for L_5^q as presented in (6)-(8).

In the next example, the two ends of the graph are joined up and each vertex is linked to two others, resulting in the cycle graph, or n-gon, C_n.

Example 2.7. Let $n \geqslant 1$ be odd and let $q \in \mathbb{K}^*$. Let C_n^q denote the $\mathbb{K}$-algebra generated by $x_1, x_2, \ldots, x_n$ subject to the relations

$$x_i x_{i+1} - q x_{i+1} x_i = 1 - q, \quad 1 \leqslant i \leqslant n-1, \tag{9}$$

$$x_n x_1 - q x_1 x_n = 1 - q, \tag{10}$$

$$x_i x_j - q x_j x_i = 0, \qquad i \geqslant 1,\ i+1 < j \leqslant n,\ j-i \text{ odd}, \tag{11}$$

$$x_i x_j - q^{-1} x_j x_i = 0, \qquad i \geqslant 1,\ i+1 < j < n,\ j-i \text{ even}. \tag{12}$$

In comparison with the odd case of L_n^q, the relation $x_1 x_n - q^{-1} x_n x_1 = 0$ is replaced by the quantized Weyl relation $x_n x_1 - q x_1 x_n = 1 - q$. As in Example 2.6, one can show that C_n^q is the skew polynomial ring $L_{n-1}^q[x_n; \tau_n, \partial_n]$ for the same $\mathbb{K}$-automorphism τ_n of L_{n-1}^q as for L_n^q and the τ_n-derivation ∂_n of L_{n-1}^q such that

$$\begin{aligned} \partial_n(x_1) &= 1 - q, \\ \partial_n(x_j) &= 0, \qquad 2 \leqslant j \leqslant n-2, \\ \partial_n(x_{n-1}) &= 1 - q^{-1}. \end{aligned}$$

As for L_n^q, the algebra C_n^q satisfies the conditions for a connected quantized Weyl algebra, the relevant graph for the presentation in (9)-(12) being the cycle graph C_n. We may refer to it as the *cyclic* connected quantized Weyl algebra C_n^q. Figure 3 shows the graph for C_5^q as presented in (9)-(12).

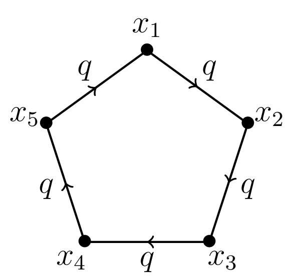

FIGURE 3. C_5^q

Remark 2.8. Cyclic connected quantized Weyl algebras with $n = 3$ are related to the quantized enveloping algebra $U_q(sl_2)$. Provided $q \neq \pm 1$, localization of $C_3^{q^2}$ at $\{x_1^i\}_{i \geq 1}$, which is a right and left Ore set by [14, Lemma 1.4], gives $U_q(sl_2)$ in the equitable presentation [21].

Example 2.9. For a $\mathbb{K}$-algebra R satisfying (i) and (ii) of Definition 2.3, neither the graph in (iii) nor its connectedness are invariants of the algebra. For example, consider the Weyl algebra A_2, generated by x_1, x_2, y_1, y_2 with relations $x_iy_i - y_ix_i = 1$, $i = 1, 2$, and commutation relations for other pairs of generators. The graph for these generators has two connected components each with two vertices and a single edge. However taking $x_1, y_1, x_1+x_2, y_1+y_2$ as generators gives a square and taking x_1, y_1, x_1+x_2, y_2 gives the path graph P_4. Thus A_2 is a connected quantized Weyl algebra although this is not apparent from its usual presentation.

A similar situation exists for higher Weyl algebras A_n, $n \geq 3$, with an increasing variety of possible graphs for different sets of generators. For example, A_3 has the hexagonal graph C_6 for the generators $x_1, y_1, x_1 + x_2, y_2, x_2 + x_3, y_1 + y_3$, the path graph P_6 for $x_1, y_1, x_2, y_2, x_2 + x_3, y_1 + y_3$ and the complete bipartite graph $K_{3,3}$ for $x_1, x_1 + x_2, x_1 + x_2 + x_3, y_1, y_1 + y_2, y_1 + y_2 + y_3$.

Whenever we refer to the graph for L_n^q or C_n^q we shall mean the graph for the presentation in (6)-(8) or (9)-(12) as appropriate.

Example 2.10. Consider the $\mathbb{K}$-algebra R generated by x_1, x_2 and x_3 subject to the relations

$$\begin{aligned} x_1x_2 - x_2x_1 &= 1, \\ x_2x_3 - x_3x_2 &= 1, \\ x_3x_1 - x_1x_3 &= 1. \end{aligned}$$

This can be obtained from C_3^q by first changing generators to replace the scalar terms $1 - q$ in the relations by 1 and then setting $q = 1$. Writing x, y and z for x_1, x_2 and x_3 respectively, R is a skew polynomial ring $A_1[z; \delta]$, where δ is the derivation of the first Weyl algebra A_1 with $\delta(x) = 1$ and $\delta(y) = -1$. As such it is a connected quantized Weyl algebra with the same graph as C_3^q. All derivations of A_1 are known to be inner [8, Lemma 4.6.8] and δ is the inner derivation induced by $-(x + y)$. Setting $t = x + y + z$, which is central, R is a polynomial ring $A_1[t]$. Relative to the generators x, y, t, the graph is not connected, having connected components $\{x, y\}$ and $\{t\}$. A similar construction reveals the polynomial algebra $A_k[t]$ over the kth Weyl algebra A_k to be a connected quantized Weyl algebra with the same cyclic graph as C_{2k+1}^q. We shall see that the algebra C_{2k+1}^q of Example 2.7 has a distinguished central element Ω which, loosely speaking, quantizes t so that the quotient $C_{2k+1}^q/\Omega C_{2k+1}^q$ quantizes A_k.

The next result tells us that, for an algebra R satisfying Condition (i) of Definition 2.3, Condition (ii) is independent of the ordering of the generators and that every algebra satisfying both is an iterated skew polynomial extension of $\mathbb{K}$.

Lemma 2.11. *Let $n \geq 2$ and, for $1 \leq i \neq j \leqslant n$, let $q_{ij} \in \mathbb{K}^*$ and $r_{ij} \in \mathbb{K}$ be such that, for $i \neq j$, $q_{ji} = q_{ij}^{-1}$ and $r_{ji} = -q_{ij}^{-1}r_{ij}$. Let R be the $\mathbb{K}$-algebra generated by $x_1, x_2, \dots, x_n$ subject to the $n(n-1)/2$ relations*

$$x_jx_i - q_{ij}x_ix_j = r_{ij}, \quad 1 \leqslant i < j \leqslant n. \tag{13}$$

The following are equivalent:

(i) *The standard monomials* $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ *form a* $\mathbb{K}$*-basis for* R*;*
(ii) *whenever* i,j,k *are distinct,* $r_{ij}\neq 0$ *implies* $q_{ik}=q_{kj}=q_{jk}^{-1}$*;*
(iii) R *is an iterated skew polynomial algebra over* $\mathbb{K}$ *in* $x_1,x_2,\dots,x_n$*;*
(iv) *for any permutation* $\sigma\in S_n$*, the standard monomials* $x_{\sigma(1)}^{a_1}x_{\sigma(2)}^{a_2}\dots x_{\sigma(n)}^{a_n}$ *form a* $\mathbb{K}$*-basis for* R*;*
(v) *for any permutation* $\sigma\in S_n$*,* R *is an iterated skew polynomial algebra over* $\mathbb{K}$ *in* $x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)}$*.*

Proof. As (ii) is invariant under permutation and (iv) and (v) are obtained from (i) and (iii) respectively by permutation, it suffices to prove the equivalence of (i), (ii) and (iii).

(i)⇒(ii). Suppose that $\{x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}\}$ is a $\mathbb{K}$-basis for R and let $1\leq u<v<w\leq n$. The element $x_wx_vx_u$ can be written in terms of this basis by computing either $(x_wx_v)x_u$ or $x_w(x_vx_u)$. Equating the resulting two expressions and cancelling the term in $x_ux_vx_w$ that appears in both gives

$$q_{vw}q_{uw}r_{uv}x_w+q_{vw}r_{uw}x_v+r_{vw}x_u=r_{uv}x_w+q_{uv}r_{uw}x_v+q_{uv}q_{uw}r_{vw}x_u.$$

If $r_{uv}\neq 0$ or, equivalently, $r_{vu}\neq 0$ then comparing coefficents of x_w gives $q_{vw}=q_{wu}$. If $r_{uw}\neq 0$, or, equivalently, $r_{wu}\neq 0$, then comparing coefficents of x_v gives $q_{vw}=q_{uv}$. If $r_{vw}\neq 0$, or, equivalently, $r_{wv}\neq 0$, then comparing coefficents of x_u gives $q_{uv}=q_{wu}$. This covers all six possibilities for the relative order of i,j,k where $\{i,j,k\}=\{u,v,w\}$ in (ii).

(ii)⇒(iii). Suppose that (ii) holds. For $1\leqslant m\leqslant n$, let R_m be the subalgebra of R generated by $x_1,x_2,\dots,x_m$. Let $i<j<k$. Consider the relation

$$x_jx_i-q_{ij}x_ix_j=r_{ij}$$

and let $x_i'=q_{ik}x_i$ and $x_j'=q_{jk}x_j$. Then

$$x_j'x_i'-q_{ij}x_i'x_j'=r_{ij}.$$

This is trivial if $r_{ij}=0$ and follows from (ii) otherwise. This gives rise, inductively, to a $\mathbb{K}$-automorphism α_k of R_{k-1} with $\alpha_k(x_i)=q_{ik}x_i$ for $1\leq i<k$. To see that there is a α_k-derivation δ_k of R_{k-1} with $\delta_k(x_i)=r_{ik}$ for $1\leq i<k$, we need to check that

$$r_{jk}x_i+\alpha(x_j)r_{ik}-q_{ij}r_{ik}x_j-q_{ij}\alpha(x_i)r_{jk}=0,$$

that is,

$$r_{jk}(1-q_{ij}q_{ik})x_i+r_{ik}(q_{jk}-q_{ij})x_j=0,$$

which is immediate from (ii). Using [9, Proposition 1], it now follows, inductively, that $R_k=R_{k-1}[x_k;\alpha_k,\delta_k]$ and hence that R is an iterated skew polynomial algebra in $x_1,x_2,\dots,x_n$ over $\mathbb{K}$.

(iii)⇒(i). This is immediate from the fact that, as a left R-module, any skew polynomial ring $R[x;\alpha,\delta]$ is free as a left R-module with basis $1,x,x^2,\dots$. □

Corollary 2.12. *Let* R *be a connected quantized Weyl algebra generated by* $x_1,x_2,\dots,x_n$ *with parameters* q_{ij} *and* r_{ij}*. There exists* $q\in\mathbb{K}^*$ *such that* $\{q_{ij}:1\leq i\neq j\leq n\}=\{q,q^{-1}\}$*.*

Proof. As R is connected and every finite connected graph has a spanning tree, we can renumber the generators so that, for $1 \leq m \leq n$, the subgraph corresponding to the subalgebra R_m generated by $x_1, \dots, x_m$ is connected. By Lemma 2.11, R_m is a connected quantized Weyl algebra for $m \geq 2$. For $2 \leq m \leq n$, let $Q_m = \{q_{ij} : 1 \leq i \neq j \leq m\}$. Then $Q_2 = \{q, q^{-1}\}$ where $q = q_{12}$. Let $2 \leq m < n$, suppose that $Q_m = \{q, q^{-1}\}$ and let $i = m + 1$. There exist j, k such that $1 \leq k < j \leq m$, $r_{ij} \neq 0$ and $r_{jk} \neq 0$. By Lemma 2.11(ii), $q_{ij} = q_{ki} = q_{jk} \in \{q, q^{-1}\}$ and $q_{i\ell} = q_{\ell j} \in \{q, q^{-1}\}$ for all $\ell \in \{1, 2, \dots, m\} \backslash \{j\}$. It follows that $Q_{m+1} = \{q, q^{-1}\}$ and, by induction, that $Q_n = \{q, q^{-1}\}$ as required. □

Corollary 2.13. *Let R be a connected quantized Weyl algebra generated by $x_1, x_2, \dots, x_n$ with single parameter q as in Corollary 2.12. Suppose that $q \neq \pm 1$. If $r_{ij} \neq 0$ and $r_{jk} \neq 0$ then $r_{j\ell} = 0$ for $\ell \in \{1, 2, \dots, n\} \backslash \{i, j, k\}$. In other words, in the graph associated with the given presentation of R, the maximum degree of a vertex cannot exceed two.*

Proof. Suppose that $r_{ij} \neq 0$, $r_{jk} \neq 0$ and $r_{j\ell} \neq 0$. Without loss of generality, assume that $q_{ij} = q$. By Lemma 2.11(ii) applied to i, j and either ℓ or k, $q_{j\ell} = q_{jk} = q$ whereas, by Lemma 2.11(ii) applied to k, j and ℓ, $q_{j\ell} = q_{kj} = q^{-1}$. As $q \neq \pm 1$, this is a contradiction. □

The next result is readily checked from the defining relations for L_n^q and C_n^q.

Proposition 2.14. *Let $n \geq 1$ and let $q \in \mathbb{K}^* \backslash \{1\}$.*

(i) *Let $\nu \in \mathbb{K}^*$. There is a $\mathbb{K}$-automorphism ι_ν of L_n^q such that $\iota_\nu(x_i) = \nu^{(-1)^i} x_i$ for $1 \leq i \leq n$.*
(ii) *If n is odd then there is a $\mathbb{K}$-automorphism of ι of C_n^q such that $\iota(x_i) = -x_i$ for $1 \leq i \leq n$.*
(iii) *There is an injective $\mathbb{K}$-homomorphism $\theta : L_{n-1}^q \to L_n^q$ such that $\theta(x_i) = x_{i+1}$ for $1 \leq i \leq n - 1$.*
(iv) *If n is odd then there is a $\mathbb{K}$-automorphism θ of C_n^q such that $\theta(x_i) = x_{i+1}$ for $1 \leq i \leq n$, where subscripts are interpreted modulo n in $\{1, 2, \dots, n\}$.*
(v) *There is a $\mathbb{K}$-isomorphism from L_n^q to $L_n^{q^{-1}}$ such that $x_i \mapsto x_{n-i+1}$ for $1 \leq i \leq n$.*
(vi) *If n is odd then there is a $\mathbb{K}$-isomorphism from C_n^q to $C_n^{q^{-1}}$ such that $x_i \mapsto x_{n-i+1}$ for $1 \leq i \leq n$.*

Remark 2.15. When n is odd and $j < n$ then L_{j-1}^q and L_j^q are subalgebras of C_n^q and the injective $\mathbb{K}$-homomorphism $\theta : L_{j-1}^q \to L_j^q$ in Proposition 2.14(iii) is the restriction to L_{j-1}^q of the automorphism θ of C_n^q in Proposition 2.14(iv).

Proposition 2.16. *Let $n \geq 2$ and let R be a connected quantized Weyl algebra generated by $x_1, x_2, \dots, x_n$ with single parameter q as in Corollary 2.12. Suppose that $q \neq \pm 1$. Then $R \simeq L_n^q$ or $R \simeq C_n^q$.*

Proof. As in the proof of Corollary 2.12, we can renumber the generators so that, for $2 \leq i \leq n$, the algebra R_i generated by $x_1, \dots, x_i$ is a connected quantized Weyl algebra. In view of Proposition 2.14(v,vi), we can assume that $q_{12} = q$. We shall show, by induction on n, that there exist $\mu_1, \dots, \mu_n \in \mathbb{K}^*$ such that if $x_i' = \mu_i x_i$ then when the defining relations of R

are written in terms of the generators x_i' they become those of L_n^q or C_n^q. This is true when $n=2$, take $\mu_1=1$ and $\mu_2=(1-q)r_{12}^{-1}$. By induction we may assume that R_{n-1} has a presentation for which the graph is the same as for L_{n-1}^q or, if n is even, C_{n-1}^q and that, for $1 \leq i < j \leq n-1$, $q_{ij} = q^{(-1)^{j-i+1}}$ and, for $1 \leq i \leq n-2$, $r_{i(i+1)} = 1-q$. All vertices in the graph for C_{n-1}^q and vertices $x_2, \dots, x_{n-1}$ in the graph representing L_{n-1}^q have vertex degree 2 so, by Corollary 2.13 and as R is connected, the graph for R_{n-1} must be the same as for L_{n-1}^q, $r_{in}=0$ for $1 < i < n-1$ and $r_{n1} \neq 0$ or $r_{n-1,n} \neq 0$ or both.

First suppose that $r_{(n-1)n} \neq 0$. As $r_{(n-1)n} \neq 0$, it follows from Lemma 2.11(ii) that, for $1 \leq i < n$, $q_{in} = q_{i(n-1)}^{-1} = q^{(-1)^{n-i+1}}$. Thus $q_{ij} = q^{(-1)^{j-i+1}}$ for $1 \leq i < j \leq n$. Now suppose also that $r_{n1}=0$. Let $x_i' = x_i$ for $1 \leq i \leq n-1$ and let $x_n' = (1-q)r_{(n-1)n}^{-1}x_n$. Then $x_i', \dots, x_n'$ generate R subject to the q-commutation relations $x_i'x_j' = q^{(-1)^{j-i+1}}x_jx_i$, $i<j-1$, and the quantized Weyl relations

$$x_i'x_{i+1}' - qx_{i+1}'x_i' = 1-q, \quad 1 \leq i \leq n-1.$$

Thus $R \simeq L_n^q$. Similar calculations with the generators ordered as $x_n, x_1, \dots, x_{n-1}$ give the same conclusion when $r_{n1} \neq 0$ and $r_{(n-1)n} = 0$.

It remains to consider the case where $r_{1n} \neq 0 \neq r_{(n-1)n}$. In this case, by Lemma 2.11(ii), $q = q_{12} = q_{2n} = q^{(-1)^{n-1}}$ so, as $q \neq \pm 1$, n must be odd. The same change of generators as above gives the same relations between the x_i' but with the q-commutation relation between x_1' and x_n' replaced by

$$x_n'x_1' - qx_1'x_n' = (1-q)\lambda,$$

where $\lambda = r_{(n-1)n}^{-1}r_{n1}$. Let $\rho \in \mathbb{K}$ be such that $\rho^2 = \lambda^{-1}$ and let $x_i'' = \rho^{(-1)^{i-1}}x_i'$ for $1 \leq i \leq n$. Then $x_1'', \dots, x_n''$ generate R subject to the q-commutation relations $x_i''x_j'' = q^{(-1)^{j-i+1}}x_j''x_i''$, when $i < j-1$ unless $i=1$ and $j=n$, and the quantized Weyl relations

$$x_i''x_{i+1}'' - qx_{i+1}''x_i'' = \rho\rho^{-1}(1-q) = 1-q, \quad 1 \leq i \leq n-1,$$

and

$$x_n''x_1'' - qx_1''x_n'' = \rho^2(1-q)\lambda = 1-q.$$

Thus $R \simeq C_n^q$ in this case.

□

Remark 2.17. Proposition 2.16 is false when $q = \pm 1$. If $q = \pm 1$ and q_{ij} is always q when $i \neq j$, then, by Lemma 2.11, one can take any connected graph G on $x_1, x_2, \dots, x_n$ and construct a connected quantized Weyl algebra using the relations $x_ix_j - qx_jx_i = r_{ij}$, $1 \leq i < j \leq n$, where $r_{ij}=1$ if there is an edge between x_i and x_j in G and $r_{ij}=0$ if there is no such edge. In [6] the authors consider the $\mathbb{K}$-algebra W_n constructed in this way when $q=-1$ and G is the complete graph.

Many ring theoretic properties of C_n^q and L_n^q follow from the fact that they are iterated skew polynomial extensions of the field $\mathbb{K}$.

Proposition 2.18. (i) The connected quantized Weyl algebras C_n^q and L_n^q are right and left noetherian domains with $\mathbb{K}^*$ as their group of units.
(ii) If M is a simple module over either C_n^q or L_n^q then $\operatorname{End} M = \mathbb{K}$. If R is a prime factor ring of either C_n^q or L_n^q for which the centre $Z(R)$ of R is not $\mathbb{K}$ then R is not primitive.
(iii) If R is a prime factor of either C_n^q or L_n^q then the Jacobson radical $\operatorname{Jac}(R) = 0$. If, further, the intersection of the non-zero prime ideals of R is non-zero then R is primitive.

Proof. (i). C_n^q and L_n^q are right and left noetherian domains by repeated application of [28, Theorem 1.2.9(iv) and (i)] or [18, Theorem 2.6 and Exercise 2O]. Although explicit references are elusive, it is well-known and easy to see, using degree, that, for a skew polynomial ring $R[x;\alpha,\delta]$ over a ring R with an automorphism α and α-derivation δ, $U(R[x;\alpha,\delta]) = U(R)$. Hence $U(C_n^q) = \mathbb{K}^* = U(L_n^q)$.
(ii) and (iii). By [28, Example 1.6.11], C_n^q and L_n^q are constructible $\mathbb{K}$-algebras in the sense of [28, 9.4.12] so, by [28, Theorem 9.4.21], they satisfy the Nullstellensatz over $\mathbb{K}$ as stated in [28, 9.1.4]. Thus every factor ring has nil Jacobson radical, and for any simple module M, $\operatorname{End} M$ is algebraic over $\mathbb{K}$. Here $\mathbb{K}$ is algebraically closed so $\operatorname{End} M = \mathbb{K}$. If $Z(R) \neq \mathbb{K}$ and $\Phi \in Z(R)\backslash\mathbb{K}$ then multiplication by Φ induces an endomorphism of any simple module M. As $\operatorname{End} M = \mathbb{K}$, M is annihilated by $\Phi - \mu$ for some $\mu \in \mathbb{K}$, whence R cannot be primitive.

As R is noetherian and prime, the nil ideal $\operatorname{Jac}(R)$ is nilpotent and hence 0. If the intersection of the non-zero prime ideals of R is non-zero then, as $\operatorname{Jac}(R) = 0$, the ideal 0 must be primitive. □

Proposition 2.19. *If q is not a root of unity then every prime ideal of L_n^q is completely prime.*

Proof. This is a consequence of [16, Theorem 2.3]. Conditions (a) and (b) of that result are clearly satisfied. Condition (c) holds because, in Example 2.6, $\tau_i\delta_i(x_j) = 0 = \delta_i\tau_i(x_j)$ for $1 \leq j < i-1$ and $\tau_i\delta_i(x_{i-1}) = 1 - q^{-1} = q\delta_i\tau_i(x_{i-1})$. Both (d) and the supplementary condition on the group Γ, which here is $\langle q \rangle$, hold because q is not a root of unity. □

Remark 2.20. We shall see that the analogue of Proposition 2.19 for C_n^q is false. The conditions of [16, Theorem 2.3] break down in a rather minimal way. When the final generator x_n is adjoined, Condition (c) fails because $\tau_n\partial_n(x_j) = 0 = \delta_n\partial_n(x_j)$ for $1 < j < n-1$ and $\tau_n\partial_n(x_{n-1}) = 1 - q^{-1} = q\partial_n\tau_n(x_{n-1})$, whereas $\tau_n\partial_n(x_1) = 1 - q = q^{-1}\partial_n\tau_n(x_1)$.

3. Prime Spectrum of L_n^q

The purpose of this section is to determine the prime spectrum of the linear connected quantized Weyl algebras L_n^q, $n \geq 3$. It is well-known, for example [14, 8.4 and 8.5] or [24, Example 6.3(iii)], that the quantized Weyl algebra A_1^q has a distinguished normal element z such that, if q is not a root of unity, the localization of A_1^q at the powers of z is simple. With A_1^q as in 2.1, $z = x_1x_2 - 1 = q(x_2x_1 - 1)$. We shall identify a sequence of elements z_i, $-1 \leq i \leq n$, such that, for $1 \leq i \leq n$, z_i is normal in L_i^q.

Notation 3.1. Let $n \geq 3$. In L_n^q, let $z_{-1} = 0$, $z_0 = 1$ and, for $1 \leq i \leq n$, let $z_i = z_{i-1}x_i - z_{i-2}$.

The next lemma gives an alternative expression for z_i in terms of the $\mathbb{K}$-homomorphism $\theta : L^q_{n-1} \to L^q_n$ of Proposition 2.14(iii), for which $\theta^2(L^q_{n-2}) \subset L^q_n$.

Lemma 3.2. *For* $1 \leq i \leq n$, $z_i = x_1\theta(z_{i-1}) - \theta^2(z_{i-2})$.

Proof. The proof is by induction on i. When $i = 1$, $x_1\theta(z_0) - \theta^2(z_{-1}) = x_1 = z_1$ and, when $i = 2$, $x_1\theta(z_1) - \theta^2(z_0) = x_1x_2 - 1 = z_2$.

If $i > 2$ and the result holds for $i-1$ and $i-2$, then

$$\begin{aligned} z_i &= z_{i-1}x_i - z_{i-2} \\ &= x_1\theta(z_{i-2})x_i - \theta^2(z_{i-3})x_i - x_1\theta(z_{i-3}) + \theta^2(z_{i-4}) \\ &= x_1\theta(z_{i-2}x_{i-1} - z_{i-3}) - \theta^2(z_{i-3}x_{i-2} - z_{i-4}) \\ &= x_1\theta(z_{i-1}) - \theta^2(z_{i-2}). \end{aligned}$$

□

We now seek relations between x_i and z_j, for $1 \leq i, j \leq n$, and between z_i and z_j when $i \neq j$.

Lemma 3.3. *Let* $1 \leq i, j \leq n$. *Then*

$$x_iz_j = \begin{cases} q^{(-1)^{i-1}} z_jx_i & \text{if } j \text{ is odd and } j < i-1, \\ z_jx_i & \text{if } j \text{ is even and } j < i-1, \\ z_{i-1}x_i + (q-1)z_{i-2} & \text{if } i \text{ is odd and } j = i-1, \\ q^{-1}z_{i-1}x_i + (1-q^{-1})z_{i-2} & \text{if } i \text{ is even and } j = i-1, \\ z_jx_i & \text{if } j \text{ is odd and } j \geq i, \\ q^{(-1)^{i-1}} z_jx_i & \text{if } j \text{ is even and } j \geq i. \end{cases}$$

Proof. This is a straightforward induction on j using the defining relations and the equations $z_k = z_{k-1}x_k - z_{k-2}$, and applying the previous two cases at each inductive step. The trivial case $j = 0$, where $z_j = 1$, can be used along with the case $j = 1$ in the initial step. Separate calculations are needed for the cases $j < i-2$, $j = i-1$, $j = i$, $j = i+1$ and $j \geq i+2$. We give details for the cases $j = i-1$, $j = i$ and $j = i+1$ when j is even. The odd cases of these are similar and the cases $j < i-2$ and $j \geq i+2$ are routine. Suppose that j is even and that the result holds for $j-1$ and $j-2$. If $j = i-1$ then

$$\begin{aligned} x_iz_{i-1} &= x_iz_{i-2}x_{i-1} - x_iz_{i-3} \\ &= z_{i-2}(x_{i-1}x_i - (1-q)) - z_{i-3}x_i \\ &= z_{i-1}x_i - (1-q)z_{i-2}. \end{aligned}$$

If $j = i$ then

$$\begin{aligned} x_iz_i &= x_iz_{i-1}x_i - x_iz_{i-2} \\ &= q^{-1}z_{i-1}x_i^2 + (1-q^{-1})z_{i-2}x_i - z_{i-2}x_i \\ &= q^{-1}z_ix_i \end{aligned}$$

and if $j = i + 1$ then

$$\begin{aligned} x_i z_{i+1} &= x_i z_i x_{i+1} - x_i z_{i-1} \\ &= q z_i (x_{i+1} x_i + (1-q)) - z_{i-1} x_i + (1-q) z_{i-2} \\ &= q z_i x_{i+1} x_i + (1-q) z_{i-1} x_i - z_{i-1} x_i \\ &= q(z_i x_{i+1} x_i - z_{i-1}) = q z_{i+1} x_i. \end{aligned}$$

□

Corollary 3.4. *For $2 \leq i \leq n$,*

$$x_i z_{i-1} = \begin{cases} q z_{i-1} x_i + (1-q) z_i & \text{if } i \text{ is odd,} \\ z_{i-1} x_i + (q^{-1} - 1) z_i & \text{if } i \text{ is even.} \end{cases}$$

Proof. By Lemma 3.3,

$$x_i z_{i-1} = \begin{cases} z_{i-1} x_i + (q-1) z_{i-2} & \text{if } i \text{ is odd,} \\ q^{-1} z_{i-1} x_i + (1 - q^{-1}) z_{i-2} & \text{if } i \text{ is even.} \end{cases}$$

The result follows by using the equation $z_{i-2} = z_{i-1} x_i - z_i$ to substitute for z_{i-2} on the right hand side. □

Corollary 3.5. (i) *For $1 \leq i \leq n$, $z_n x_i = \rho_i x_i z_n$, where $\rho_i = 1$ if n is odd and $\rho_i = q^{(-1)^i}$ if n is even. Consequently, the element z_n is normal in L_n^q.*

(ii) *For $0 \leq i < j \leq n$, $z_i z_j = q^{\lambda_{ij}} z_j z_i$, where $\lambda_{ij} = 0$ if j is odd or if j and i are both even, and $\lambda_{ij} = 1$ if j is even and i is odd.*

Proof. (i) is immediate from Lemma 3.3 and (ii) follows inductively using the formula $z_i = z_{i-1} x_i - z_{i-2}$. □

Notation 3.6. Let $n \geq 2$, let $q \in \mathbb{K}^*$, let $\Lambda = (\lambda_{ij})$ be the $n \times n$ antisymmetric matrix over $\mathbb{K}$ such that, for $1 \leq i < j \leq n$, λ_{ij} is as specified in Lemma 3.5(ii) and, for $1 \leq i, j \leq n$, let $q_{ij} = q^{\lambda_{ij}}$. Thus $q_{ji} = q_{ij}^{-1}$, $q_{ii} = 1$ and, for $i < j$, $q_{ij} = 1$ if j is odd or if j and i are both even, and $q_{ij} = q$ if j is even and i is odd.

Let W_n^q and T_n^q, respectively, denote the co-ordinate ring of quantum n-space with generators $z_1, \ldots, z_n$ and the co-ordinate ring of the quantum n-torus with generators $z_1^{\pm 1}, \ldots, z_n^{\pm 1}$, subject to the relations $z_i z_j = q_{ij} z_j z_i$ for $1 \leq i, j \leq n$. We may also refer occasionally to the subalgebras W_i^q and T_i^q generated by $z_1, z_2, \ldots, z_i$ or by $z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_i^{\pm 1}$ as appropriate, where $1 < i < n$. As is well-known, W_n^q is an iterated skew polynomial algebra over $\mathbb{K}$, and T_n^q is an iterated skew Laurent polynomial algebra over $\mathbb{K}$.

Lemma 3.7. *Suppose that q is not a root of unity. If n is even then T_n^q is simple and if n is odd then T_n^q is a Laurent polynomial ring $T_{n-1}^q[z_n^{\pm 1}]$ over the simple ring T_{n-1}^q.*

Proof. If n is even we apply the criterion given by [27, Proposition 1.3]. Let $m_1, \ldots, m_n \in \mathbb{Z}$ be such that $q_{1j}^{m_1} q_{2j}^{m_2} \ldots q_{nj}^{m_n} = 1$ for all j, $1 \leq j \leq n$. If j is even then $q_{ij} = 1$ unless i is odd and $i < j$, in which case $q_{ij} = q$. So successive considerations of the cases $j = 2, 4, \ldots, n$

gives $m_2 = m_4 = \cdots = m_n = 0$. If j is odd then $q_{ij} = 1$ unless i is even and $i > j$, in which case $q_{ij} = q^{-1}$. Successive considerations of the cases $j = n-1, \dots, 3, 1$ gives $m_{n-1} = \cdots = m_3 = m_1 = 0$. By [27, Proposition 1.3], T_n^q is simple. If n is odd then it is clear from Corollary 3.5(ii) that $T_n^q = T_{n-1}^q[z_n^{\pm 1}]$. □

The following result of Wexler-Kreindler [31, Proposition 2] on changing the indeterminate of a skew polynomial ring will be useful.

Lemma 3.8. *Let α be an automorphism of a ring R and let δ be an α-derivation of R. Let $a \in R$ and let u be a unit in R with inner automorphism $\gamma_u : r \mapsto uru^{-1}$. Let $\alpha' = \gamma_u\alpha$ and, for $r \in R$, let $\delta'(r) = u\delta(r) + ar - \gamma_u(\alpha(r))a$. Then δ' is an α'-derivation of R and $R[x; \alpha, \delta] = R[x'; \alpha', \delta']$, where $x' = ux + a$.*

The next result allows us to identify L_n^q with an intermediate $\mathbb{K}$-algebra between W_n^q and T_n^q. After the proof of (i) this identification, which is pre-empted by the use of the notation z_i in all three algebras, will be made implicitly.

Notation 3.9. For $1 \leq j \leq n$, let $\mathcal{Z}_j$ denote the multiplicatively closed set $\{f z_1^{a_1} z_2^{a_2} \dots z_j^{a_j} : f \in \mathbb{K}^*, a_i \in \mathbb{N}_0, 1 \leq i \leq j\}$ of W_n^q. We will make use of the fact that each $\mathcal{Z}_n$ is right and left Ore and that T_n^q is the localization of W_n^q at $\mathcal{Z}_n$.

Proposition 3.10. *Let $n \geq 2$ and $q \in \mathbb{K}^*$. Suppose that q is not a root of unity.*

(i) *There are injective $\mathbb{K}$-algebra homomorphisms $\phi : W_n^q \hookrightarrow L_n^q$ and $\psi : L_n^q \hookrightarrow T_n^q$ such that $\psi\phi(z_i) = z_i$ for $1 \leq i \leq n$. (This allows us to regard W_n^q as a subalgebra of L_n^q and to regard each $\mathcal{Z}_j$ as a subset of L_n^q.)*

(ii) *The sets $\mathcal{Z}_n$ and $\mathcal{Z}_{n-1}$ are right and left Ore sets in L_n^q. The localization of L_n^q at $\mathcal{Z}_n$ is T_n^q. If n is odd then the localization of L_n^q at $\mathcal{Z}_{n-1}$ is the polynomial algebra $T_{n-1}^q[z_n]$ and if n is even it is the skew polynomial algebra $T_{n-1}^q[z_n; \alpha]$, where $\alpha(z_i) = z_i$ if i is even and $\alpha(z_i) = q^{-1}z_i$ if i is odd.*

Proof. (i) By Corollary 3.5(ii), applied to each of the algebras L_i^q, there is a $\mathbb{K}$-algebra homomorphism $\phi : W_n^q \to L_n^q$ such that $\phi(z_i) = z_i$ for $1 \leq i \leq n$. As L_n^q is a domain so also is the image $\phi(W_n^q)$. Hence $\ker\phi$ is a completely prime ideal of W_n^q. If $\ker\phi \neq 0$ then, by Lemma 3.7, either $z_i \in \ker\phi$ for some i, $1 \leq i \leq n-1$ or $z_n - \lambda \in \ker\phi$ for some $\lambda \in \mathbb{K}$ (with $\lambda = 0$ if n is even). As L_n^q has a PBW basis and the coefficient of $x_1x_2 \dots x_i$ in z_i is 1, it follows that $\ker\phi = 0$ and hence that ϕ is injective.

With $z_{-1} = 0$ and $z_0 = 1$, let $w_i = z_{i-1}^{-1}z_i$ and $v_i = w_i + w_{i-1}^{-1} = z_{i-1}^{-1}(z_i + z_{i-2}) \in T_n^q$, $1 \leq i \leq n$. It follows, by Corollary 3.5(ii), that, for $1 \leq i < j \leq n$,

$$w_j w_i = \begin{cases} q^{-1}w_iw_j & \text{if } i-j \text{ is even,} \\ qw_jw_i & \text{if } i-j \text{ is odd.} \end{cases}$$

From this it follows routinely that if $i > j+1$ then

$$v_jv_i = \begin{cases} q^{-1}v_jv_i & \text{if } i-j \text{ is even,} \\ qv_jv_i & \text{if } i-j \text{ is odd.} \end{cases}$$

and that
$$v_{i+1}v_i = qv_iv_{i+1} + 1 - q.$$
Thus the v_i's satisfy the defining relations for L_n^q and there is a $\mathbb{K}$-algebra homomorphism $\psi : L_n^q \to T_n^q$ such that $\psi(x_i) = v_i$ and $\psi(z_i) = z_i$ for $1 \leq i \leq n$. As T_n^q is a domain so also is $\psi(L_n^q)$ so $\ker \psi$ is a completely prime ideal of L_n^q.

Let $D = \{r \in L_n^q : zr \in W_n^q \text{ for some } z \in \mathcal{Z}_{n-1}\}$. Using Corollary 3.5(ii), it is easy to see that D is a subalgebra of L_n^q. For $1 \leq i \leq n$, $z_{i-1}x_i = z_i + z_{i-2}$ so each $x_i \in D$ and therefore $L_n^q = D$. Now suppose that $\ker \psi \neq 0$ and let $0 \neq f \in \ker \psi$. As $f \in D$, there exists $z \in \mathcal{Z}_{n-1}$ such that $zf \in \ker \psi \cap W_n^q$. As $\psi(z_i) = z_i$ for all i, $\ker \psi \cap W_n^q = 0$ so $zf = 0$. But L_n^q is a domain so $f = 0$. Hence $\ker \psi = 0$ and ψ is injective.

(ii) As T_n^q is a right and left quotient ring of W_n^q with respect to $\mathcal{Z}_n$, it is also a right and left quotient ring of the intermediate ring L_n^q with respect to $\mathcal{Z}_n$ and hence $\mathcal{Z}_n$ is right and left Ore in L_n^q. The set $\mathcal{Z}_{n-1}$ is a τ_n-invariant right and left Ore set in L_{n-1}^q, with quotient ring T_{n-1}^q, so it follows easily from [14, Lemma 1.4] that it is right and left Ore in L_n^q with quotient ring $T_{n-1}^q[x_n; \tau_n, \delta_n]$. As $z_n = z_{n-1}x_n - z_{n-2}$ and z_{n-1} is invertible in T_{n-1}^q, it follows from Lemma 3.8 that if n is odd then $T_{n-1}^q[x_n; \tau_n, \delta_n] = T_{n-1}^q[z_n]$ and that if n is even it is the skew polynomial algebra $T_{n-1}^q[x_n; \tau_n, \delta_n] = T_{n-1}^q[z_n; \tau]$ where $\tau(z_i) = z_i$ if i is even and $\tau(z_i) = q^{-1}z_i$ if i is odd. □

Notation 3.11. For $m \in \mathbb{N}$ and $q \in \mathbb{K}^*$, let $[m]_q := 1 + q + q^2 + \ldots + q^{m-1}$, which, if $q \neq 1$, is $(q^m - 1)/(q - 1)$.

Proposition 3.12. *Let $n \geq 2$ and let $z = fz_1^{a_1}z_2^{a_2}\ldots z_{n-1}^{a_{n-1}} \in \mathcal{Z}_{n-1}$, where $f \in \mathbb{K}^*$ and $a_i \in \mathbb{N}_0$ for $1 \leq i \leq n-1$. Let $L = L_n^q$ and suppose that q is not a root of unity. Then $LzL = L$ and $P \cap \mathcal{Z}_{n-1} = \emptyset$ for all prime ideals P of L.*

Proof. Suppose that $LzL \neq L$ and let M be a maximal ideal of L containing LzL. By Proposition 2.19, M is completely prime and so $z_i \in M$ for some i, $1 \leq i \leq n-1$. By Lemma 3.3, $z_j \in M$ for $0 \leq j \leq i-1$. In particular, $1 = z_0 \in M$ which is impossible. Hence $LzL = L$ and consequently $P \cap \mathcal{Z}_{n-1} = \emptyset$ for all prime ideals P of L. □

We are now in a position to determine the prime ideals of L_n^q when q is not a root of unity.

Theorem 3.13. *Let $L = L_n^q$ and suppose that q is not a root of unity. If n is odd then the prime ideals of L are 0 and, for each $\lambda \in \mathbb{K}$, $P_\lambda := (z_n - \lambda)L$. If n is even then the prime ideals of L are 0, z_nL and, for each $\lambda \in \mathbb{K}^*$, $P'_\lambda := (z_{n-1} - \lambda)L + z_nL$.*

Proof. First note that if n is odd then P_λ is an ideal of L as z_n is central. If n is even, then z_nL is an ideal of L, by Corollary 3.5(i), and z_{n-1} is central modulo $z_nL_n^q$ in L_n^q, by Corollary 3.4. So P'_λ is an ideal of L.

For $\lambda \in \mathbb{K}$, note that $z_n - \lambda = z_{n-1}x_n - z_{n-2} - \lambda$ has degree one in x_n. Using [22, Proposition 1] and Corollary 3.5, it is easily shown by induction that $(z_n - \lambda)L$ is completely prime if n is odd or if n is even and $\lambda = 0$. If $\lambda \neq 0$, $x_n \equiv \lambda^{-1}z_{n-2} \bmod P'_\lambda$, so L/P'_λ is generated by the images $\overline{x_i}$, $1 \leq i \leq n-1$, and there are inverse homomorphisms $\varphi : L/P'_\lambda \to L_{n-1}^q/(z_{n-1} - \lambda)L_{n-1}^q$ and $\theta : L_{n-1}^q/(z_{n-1} - \lambda)L_{n-1}^q \to L/P'_\lambda$ such that $\varphi(\overline{x_i}) = \overline{x_i}$

and $\theta(\overline{x_i}) = \overline{x_i}$ for $1 \leq i \leq n-1$. Thus $L/P'_\lambda \simeq L^q_{n-1}/(z_{n-1}-\lambda)L^q_{n-1}$ and hence P'_λ is also completely prime.

Let T be the localization of L at $\mathcal{Z}_{n-1}$ and let P be a non-zero prime ideal of L. Suppose that n is odd. By Proposition 3.10 and Lemma 3.7, T is the polynomial ring $T^q_{n-1}[z_n] \simeq T^q_{n-1} \otimes_{\mathbb{K}} \mathbb{K}[z_n]$ over the simple ring T^q_{n-1}. By [27, Proposition 1.3], the centre of T^q_{n-1} is $\mathbb{K}$. It is then a consequence of [28, Lemma 9.6.9(i)] that $PT = P_\lambda T$ for some $\lambda \in \mathbb{K}$. By Proposition 3.12 and [28, Proposition 2.1.16(vii)], $P = P_\lambda$.

Now suppose that n is even. By Proposition 3.10 and Lemma 3.7, T is a skew polynomial ring $T^q_{n-1}[z_n;\alpha]$ and $T^q_{n-1}[z_n^{\pm 1};\alpha]$ is simple. It follows that PT contains z_n and hence has the form $Q + z_nT$ for some prime ideal Q of $T^q_{n-1} = T^q_{n-2}[z^{\pm 1}_{n-1}]$. By [28, Lemma 9.6.9(i)], every such prime ideal of T^q_{n-1} has the form 0 or $(z_{n-1}-\lambda)T^q_{n-1}$ for some $\lambda \in \mathbb{K}^*$ so $PT = z_nT$ or $PT = P'_\lambda T$. The result in the even case now follows as in the odd case using [28, Proposition 2.16(vii)] and Proposition 3.12.

□

Recall from [7] that a noetherian domain R is called a unique factorization domain (UFD) if every non-zero prime ideal of R contains a non-zero completely prime ideal of the form pR where p is normal in R. The following is immediate from Theorem 3.13.

Corollary 3.14. *Suppose that q is not a root of unity. If $n \geq 2$ then the linear connected quantized Weyl algebra L^q_n is a UFD. If n is odd then every height one prime ideal of L^q_n is maximal. If n is even then there is a unique height one prime ideal $z_nL^q_n$ and $L^q_n/z_nL^q_n$ is a UFD in which every height one prime ideal is maximal.*

Corollary 3.15. *Let $L = L^q_n$ and suppose that q is not a root of unity. If n is odd then the primitive ideals of L are the prime ideals $(z_n-\lambda)L$, $\lambda \in \mathbb{K}$. If n is even then the primitive ideals of L are 0 and, for each $\lambda \in \mathbb{K}^*$, $(z_{n-1}-\lambda)L + z_nL$.*

Proof. Suppose that n is odd. The ideal 0 of L is not primitive, by Proposition 2.18(ii), due to the existence of the central element z_n. All the non-zero prime ideals are maximal and hence primitive.

Now suppose that n is even. The ideal 0 of L is primitive, by Proposition 2.18(iii), because L has a unique height one prime ideal. The ideal z_nL is not primitive, by Proposition 2.18(ii), as z_{n-1} is central modulo z_nL. The remaining prime ideals are maximal and hence primitive. □

4. Prime Spectrum of C^q_n

Throughout this section, $n \geq 3$ is an odd positive integer. Recall that the cyclic connected quantized Weyl algebra C^q_n has the form $L^q_{n-1}[x_n;\tau_n,\partial_n]$ and contains L^q_m as a subalgebra for $1 \leq m \leq n-1$. Let θ be the $\mathbb{K}$-automorphism of C^q_n specified in Proposition 2.14(iii). Thus $\theta(x_i) = x_{i+1}$ for $1 \leq i \leq n-1$ and $\theta(x_n) = x_1$. For $1 \leq m < n$, $\theta(L^q_m)$ is the $\mathbb{K}$-subalgebra of C^q_n generated by $x_2, x_3, \ldots, x_{m+1}$. The relations that the elements $z_1, z_2, \ldots, z_{n-1} \in C^q_n$ satisfy with each other, and with $x_1, x_2, \ldots, x_{n-1}$, are as in Lemma 3.3 and Corollaries 3.4 and 3.5. The next lemma gives the relations between x_n and each of $z_1, z_2, \ldots, z_{n-1}$ in

C_n^q and, by modifying the formula that defined z_n in L_n^q, identifies a distinguished central element of C_n^q.

Lemma 4.1. *Let $\Omega = z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2}) \in C_n^q$ and let θ be the $\mathbb{K}$-automorphism of C_n^q specified in Proposition 2.14(iii).*

(i) *For $1 \leq j \leq n-2$,*

$$x_n z_j = \begin{cases} qz_j x_n + (1-q)\theta(z_{j-1}) & \text{if } j \text{ is odd,} \\ z_j x_n + (1-q)\theta(z_{j-1}) & \text{if } j \text{ is even.} \end{cases}$$

(ii) $x_n z_{n-1} = z_{n-1}x_n + (1-q)(\theta(z_{n-2}) - z_{n-2})$.

(iii) $\theta(\Omega) = \Omega$.

(iv) Ω *is central in* C_n^q.

Proof. Recall that $C_n^q = L_{n-1}^q[x_n; \tau_n, \partial_n]$ for the $\mathbb{K}$-automorphism τ_n of L_{n-1}^q such that $\tau_n(x_j) = q^{(-1)^{j-1}}x_j$ for $1 \leq j \leq n-1$ and the τ_n-derivation ∂_n of L_{n-1}^q such that $\partial_n(x_1) = 1-q$, $\partial_n(x_j) = 0$ for $2 \leq j \leq n-2$ and $\partial_n(x_{n-1}) = 1 - q^{-1}$.

For (i), let $1 \leq j \leq n-2$. It is a routine matter to check, by induction, that $\tau_n(z_j) = qz_j$ if j is odd and $\tau_n(z_j) = z_j$ if j is even and that $\partial_n(z_j) = \partial_n(z_{j-1}x_j - z_{j-2}) = (1-q)\theta(z_{j-1})$. The result (i) follows.

Using (i), we see that $\tau_n(z_{n-1}) = \tau_n(z_{n-2}x_{n-1} - z_{n-3}) = z_{n-1}$ and that

$$\begin{aligned} \partial_n(z_{n-1}) &= \partial_n(z_{n-2}x_{n-1} - z_{n-3}) \\ &= (1-q)\theta(z_{n-3})\theta(x_{n-2}) + (1-q^{-1})qz_{n-2} - (1-q)\theta(z_{n-4}) \\ &= (1-q)(\theta(z_{n-2}) - z_{n-2}). \end{aligned}$$

Thus (ii) holds. For (iii),

$$\begin{aligned} \theta(\Omega) &= \theta(z_{n-1}x_n) - \theta(z_{n-2}) - q\theta^2(z_{n-2}) \\ &= \theta(x_n z_{n-1} + (1-q)(z_{n-2} - \theta(z_{n-2}))) - \theta(z_{n-2}) - q\theta^2(z_{n-2}) && \text{(by (ii))} \\ &= x_1\theta(z_{n-1}) - q\theta(z_{n-2}) - \theta^2(z_{n-2}) \\ &= x_1\theta(z_{n-2})x_n - x_1\theta(z_{n-3}) - \theta^2(z_{n-3})x_n + \theta^2(z_{n-4}) - q\theta(z_{n-2}) \\ &= z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2}) && \text{(by 3.2)} \\ &= \Omega. \end{aligned}$$

For (iv), first note that, by Lemma 3.3, $x_{n-1}z_{n-2} = q^{-1}z_{n-2}x_{n-1} + (1-q^{-1})z_{n-3}$ so that

$$qx_n\theta(z_{n-2}) = \theta(qx_{n-1}z_{n-2}) = \theta(z_{n-2}x_{n-1} + (q-1)z_{n-3}). \tag{14}$$

By (ii), (i) and (14),

$$\begin{aligned}
x_n\Omega &= x_n(z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2}))\\
&= (z_{n-1}x_n + (1-q)(\theta(z_{n-2}) - z_{n-2}))x_n - qz_{n-2}x_n - (1-q)\theta(z_{n-3}) - qx_n\theta(z_{n-2})\\
&= z_{n-1}x_n^2 + (1-q)\theta(z_{n-2})x_n - z_{n-2}x_n - (1-q)\theta(z_{n-3}) - qx_n\theta(z_{n-2})\\
&= z_{n-1}x_n^2 - z_{n-2}x_n - q\theta(z_{n-2})x_n\\
&= \Omega x_n.
\end{aligned}$$

By (iii), $x_i\Omega = \Omega x_i$ for $1 \le i \le n$, so Ω is central in C_n^q. □

Proposition 4.2. *Let $q \in \mathbb{K}^*$. Suppose that q is not a root of unity. The subsets $\mathcal{Z}_{n-1}$ and $\mathcal{Z}_{n-2}$ of L_{n-1}^q are right and left Ore sets in C_n^q. The localization of C_n^q at $\mathcal{Z}_{n-1}$ is the polynomial ring $T_{n-1}^q[\Omega]$ over the simple algebra T_{n-1}^q.*

Proof. By Proposition 3.10, $\mathcal{Z}_{n-1}$ is a right and left Ore set in L_{n-1}^q and it is clearly τ_n-invariant. By [14, Lemma 1.4], $\mathcal{Z}_{n-1}$ is right and left Ore in C_n^q with quotient ring $T_{n-1}^q[x_n;\tau_n,\partial_n]$. Similarly, $\mathcal{Z}_{n-2}$ is a right and left Ore set in L_{n-1}^q and in C_n^q.

As $\Omega = ux_n + a$, where $u = z_{n-1}$, which is invertible in T_{n-1}^q, and $a = -(z_{n-2} + q\theta(z_{n-2})) \in L_{n-1}^q$, it follows from Lemma 3.8 that $T_{n-1}^q[x_n;\tau_n,\partial_n]$ has the form $T_{n-1}^q[\Omega;\tau_n',\partial_n']$. As Ω is central, τ_n' must be the identity automorphism on L_{n-1}^q and ∂_n' must be the zero derivation. □

The next lemma will be significant in identifying the localization of C_n^q at $\mathcal{Z}_{n-2}$ as an *ambiskew polynomial ring*. Our notation for such rings and the related *generalized Weyl algebras* will be essentially as in [24]. Given a $\mathbb{K}$-algebra A, commuting $\mathbb{K}$-automorphisms α and γ of A, an element $v \in A$ such that $va = \gamma(a)v$ for all $a \in A$ and $\gamma(v) = v$, and a scalar $\rho \in \mathbb{K}\backslash\{0\}$, the ambiskew polynomial ring $R = R(A,\alpha,v,\rho)$ is the iterated skew polynomial ring $A[y;\alpha][x;\beta,\delta]$, where $\beta = \alpha^{-1}\gamma$ is extended to a $\mathbb{K}$-automorphism of $A[y;\alpha]$ by setting $\beta(y) = \rho y$ and δ is a β-derivation of $A[y;\alpha]$ such that $\delta(A) = 0$ and $\delta(y) = v$. Thus $xy = \rho yx + v$ and, for all $a \in A$, $ya = \alpha(a)y$ and $xa = \beta(a)x$.

If v is regular, as will be the case in all examples considered here, then v determines γ.

If there exists $u \in A$ such that $ua = \gamma(a)u$ for all $a \in A$ and $v = u - \rho\alpha(u)$ then the element $z := xy - u = \rho(yx - \alpha(u))$ is such that $zy = \rho yz$, $zx = \rho^{-1}xz$, $za = \gamma(a)z$ for all $a \in A$ and $zu = uz$. If such an element u exists then R is a *conformal ambiskew polynomial ring*, u is a *splitting element* and z, which is normal in R, is the corresponding *Casimir element* of R. The factor R/zR, which we denote here by $W(A,\alpha,u)$, is then generated by A, $X := x + zR$ and $Y := y + zR$ subject to the relations $XY = u$, $YX = \alpha(u)$ and, for all $a \in A$, $Ya = \alpha(a)Y$ and $Xa = \beta(a)X$. In the case where u is central this is one of the algebras named *generalized Weyl algebras* in [1] and we use the same name here.

Lemma 4.3. *For $q \in \mathbb{K}^*$ and $1 \le i \le n-1$,*

$$\begin{aligned}
qz_i\theta(z_{i-2}) - z_{i-1}\theta(z_{i-1}) &= -q^{\frac{i-1}{2}} \quad \textit{if } i \textit{ is odd and}\\
z_i\theta(z_{i-2}) - z_{i-1}\theta(z_{i-1}) &= -q^{\frac{i-2}{2}} \quad \textit{if } i \textit{ is even.}
\end{aligned}$$

Proof. When $i = 1$, $z_{i-2} = 0$ and $z_{i-1} = 1$ so the result holds. Let $i > 1$ and suppose that the result holds for $i - 1$. If i is odd then

$$\begin{aligned}
&qz_i\theta(z_{i-2}) - z_{i-1}\theta(z_{i-1}) \\
&= q(z_{i-1}x_i - z_{i-2})\theta(z_{i-2}) - z_{i-1}\theta(z_{i-2}x_{i-1} - z_{i-3}) \\
&= z_{i-1}\theta(qx_{i-1}z_{i-2} - z_{i-2}x_{i-1} + z_{i-3}) - qz_{i-2}\theta(z_{i-2}) \\
&= qz_{i-1}\theta(z_{i-3}) - qz_{i-2}\theta(z_{i-2}) && \text{(by 3.3)} \\
&= -q^{\frac{i-1}{2}}.
\end{aligned}$$

If i is even then

$$\begin{aligned}
&z_i\theta(z_{i-2}) - z_{i-1}\theta(z_{i-1}) \\
&= (z_{i-1}x_i - z_{i-2})\theta(z_{i-2}) - z_{i-1}\theta(z_{i-2}x_{i-1} - z_{i-3}) \\
&= z_{i-1}\theta(x_{i-1}z_{i-2} - z_{i-2}x_{i-1} + z_{i-3}) - z_{i-2}\theta(z_{i-2}) \\
&= qz_{i-1}\theta(z_{i-3}) - z_{i-2}\theta(z_{i-2}) && \text{(by 3.3)} \\
&= -q^{\frac{i-2}{2}}.
\end{aligned}$$

The result follows by induction on i. □

Proposition 4.4. *Let $n \geq 3$ be odd and let $q \in \mathbb{K}^*$. Suppose that q is not a root of unity. Let α be the $\mathbb{K}$-automorphism of T^q_{n-2} such that, for $1 \leq i \leq n-2$, $\alpha(z_i) = z_i$ if i is even and $\alpha(z_i) = q^{-1}z_i$ if i is odd.*

(i) *The localization S of C^q_n at $\mathcal{Z}_{n-2}$ is the ambiskew polynomial algebra $R(T^q_{n-2}, \alpha, v, 1)$ where $v = (1-q)(q^{\frac{n-3}{2}}z^{-1}_{n-2} - z_{n-2})$, $x = \theta^{-1}(z_{n-1})$ and $y = z^{-1}_{n-2}z_{n-1}$.*

(ii) *For any $\lambda \in \mathbb{K}$, the element $q^{\frac{n-3}{2}}z^{-1}_{n-2} + \lambda + qz_{n-2}$ is splitting, $\Omega - \lambda$ is a central Casimir element and $S/(\Omega - \lambda)S$ is a generalized Weyl algebra over T^q_{n-2}.*

Proof. (i) By [14, Lemma 1.4], $T^q_{n-2}[x_{n-1}; \tau_{n-1}, \partial_{n-1}][x_n; \tau_n, \partial_n]$ is the localization of C^q_n at $\mathcal{Z}_{n-2}$. Observe that $y = x_{n-1} - z^{-1}_{n-2}z_{n-3}$ and that, by Lemma 3.2,

$$\begin{aligned}
x = \theta^{-1}(z_{n-1}) &= x_nz_{n-2} - \theta(z_{n-3}) \\
&= qz_{n-2}x_n + (1-q)\theta(z_{n-3}) - \theta(z_{n-3}) \text{ (by 4.1(i))} \\
&= q(z_{n-2}x_n - \theta(z_{n-3})).
\end{aligned}$$

As qz_{n-2} is a unit in T^q_{n-2}, it follows from Lemma 3.8 that S is an iterated skew polynomial ring of the form $T^q_{n-2}[y; \tau'_{n-1}, \partial'_{n-1}][x; \tau'_n, \partial'_n]$ over T^q_{n-2}. By Corollary 3.5, $yx_i = q^{(-1)^i}x_iy$ for $1 \leq i \leq n-2$ and it follows that $\tau'_{n-1} = \alpha$ and $\partial'_{n-1} = 0$. Also by Corollary 3.5, $z_{n-1}x_{i+1} = q^{(-1)^{i+1}}x_{i+1}z_{n-1}$ for $1 \leq i \leq n-2$. Applying θ^{-1}, we see that $xx_i = q^{(-1)^{i+1}}x_ix$ for $1 \leq i \leq n-2$ and hence that the restrictions of τ'_n and ∂'_n to T^q_{n-2} are α^{-1} and 0 respectively. It remains to show that $\tau'_n(y) = y$ and that $\partial'_n(y) = v$.

By Lemma 4.1(i), $\tau_n(z_{n-1}) = z_{n-1}$ and $\tau_n(z_{n-2}) = qz_{n-2}$ and, by 3.5, $\gamma_{qz_{n-2}}(z_{n-1}) = qz_{n-1}$. It follows, by Lemma 3.8, that $\tau'_n(y) = y$.

We have seen that $\partial'_n(t) = 0$ for all $t \in T^q_{n-2}$, so $\partial'_n(y) = \partial'_n(x_{n-1} - z^{-1}_{n-2}z_{n-3}) = \partial'_n(x_{n-1})$. Applying θ^{-1} to the equation in Lemma 4.1(ii), we obtain

$$xx_{n-1} = x_{n-1}x + (q-1)(z_{n-2} - \theta^{-1}(z_{n-2})).$$

Here $\theta^{-1}(z_{n-2})) \notin T^q_{n-2}[y; \tau'_{n-1}, \partial'_{n-1}]$ but, using Lemmas 3.2 and 4.1(i),

$$\begin{aligned}\theta^{-1}(z_{n-2}) &= x_n z_{n-3} - \theta(z_{n-4})\\ &= z_{n-3}x_n - q\theta(z_{n-4})\\ &= z_{n-3}(q^{-1}z^{-1}_{n-2}(x + q\theta(z_{n-3})) - q\theta(z_{n-4})\end{aligned}$$

so

$$xx_{n-1} = (x_{n-1} + (q^{-1}-1)z_{n-3}z^{-1}_{n-2})x + (q-1)(z_{n-2} + q\theta(z_{n-4}) - z_{n-3}z^{-1}_{n-2}\theta(z_{n-3})).$$

As $x_{n-1} + (q^{-1}-1)z_{n-3}z^{-1}_{n-2}$ and $(q-1)(z_{n-2} + q\theta(z_{n-4}) - z_{n-3}z^{-1}_{n-2}\theta(z_{n-3}))$ are both in $T^q_{n-2}[y; \tau'_{n-1}, \partial'_{n-1}]$, and as z_{n-2} is central in T^q_{n-2}, it follows that

$$\begin{aligned}\partial'_n(y) = \partial'_n(x_{n-1}) &= (q-1)(z_{n-2} + q\theta(z_{n-4}) - z^{-1}_{n-2}z_{n-3}\theta(z_{n-3}))\\ &= (q-1)(z_{n-2} + z^{-1}_{n-2}(qz_{n-2}\theta(z_{n-4}) - z_{n-3}\theta(z_{n-3})))\\ &= (q-1)(z_{n-2} - q^{\frac{n-3}{2}}z^{-1}_{n-2}) \qquad \text{(by Lemma 4.3)}\\ &= v.\end{aligned}$$

This completes the proof of (i).

(ii) Let $\lambda \in \mathbb{K}$ and let $u = q^{\frac{n-3}{2}}z^{-1}_{n-2} + \lambda + qz_{n-2}$. Then u is central in T^q_{n-2} because z_{n-2} is central. Also

$$\begin{aligned}u - \alpha(u) &= q^{\frac{n-3}{2}}z^{-1}_{n-2} + \lambda + qz_{n-2} - qq^{\frac{n-3}{2}}z^{-1}_{n-2} - \lambda - z_{n-2}\\ &= (q-1)(z_{n-2} - q^{\frac{n-3}{2}}z^{-1}_{n-2})\\ &= v\end{aligned}$$

so u is a splitting element.

As z_{n-2} is central in L^q_{n-2}, the corresponding Casimir element is

$$z := xy - u = \theta^{-1}(z_{n-1})(x_{n-1} - z_{n-3}z^{-1}_{n-2}) - q^{\frac{n-3}{2}}z^{-1}_{n-2} - \lambda - qz_{n-2}.$$

Recall that $\Omega = z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2}) \in C^q_n$ and, from Lemma 4.1(iii), that $\theta(\Omega) = \Omega$. Hence $\theta^{-1}(z_{n-1})x_{n-1} = \Omega + \theta^{-1}(z_{n-2}) + qz_{n-2}$ and it follows that

$$z = \Omega - \lambda + \theta^{-1}(z_{n-2}) - \theta^{-1}(z_{n-1})z_{n-3}z^{-1}_{n-2} - q^{\frac{n-3}{2}}z^{-1}_{n-2}.$$

By Lemma 4.3 with $i = n-2$, $z_{n-2}\theta(z_{n-2}) = z_{n-1}\theta(z_{n-3}) + q^{\frac{n-3}{2}}$ so, applying θ^{-1} and postmultiplying by z^{-1}_{n-2},

$$\theta^{-1}(z_{n-2}) = \theta^{-1}(z_{n-1})z_{n-3}z^{-1}_{n-2} + q^{\frac{n-3}{2}}z^{-1}_{n-2}.$$

Hence $z = \Omega - \lambda$ is a Casimir element and $S/(\Omega - \lambda)S$ is a generalized Weyl algebra over T^q_{n-2}. $\square$

Proposition 4.5. *There is a bijection between* $\operatorname{Spec} C_n^q$ *and* $\operatorname{Spec} S$ *given by* $P \mapsto PS$, *for* $P \in \operatorname{Spec} C_n^q$, *and* $Q \mapsto Q \cap C_n^q$, *for* $Q \in \operatorname{Spec} S$.

Proof. For all $z \in \mathcal{Z}_{n-2}$, $L_{n-1}^q z L_{n-1}^q = L_{n-1}^q$, by Proposition 3.12, and so $C_n^q z C_n^q = C_n^q$. Hence $P \cap \mathcal{Z}_{n-2} = \emptyset$ for all $P \in \operatorname{Spec} C_n^q$. The result follows by [28, Proposition 2.1.16(vii)]. □

The ambiskew polynomial ring S is the main example of [10], where its prime spectrum is computed. It consists of

- 0;
- a height one prime ideal $(\Omega - \lambda)S$ for each $\lambda \in \mathbb{K}$;
- for each positive integer m, two maximal ideals $F_{m,1}$ and $F_{m,-1}$ such that $S/F_{m,1}$ and $S/F_{m,-1}$ have Goldie rank m.

Here $F_{m,1}$ contains the height one prime ideal $(\Omega - \lambda)S$, where $\lambda = (q^m + 1)q^{\frac{n-2m-1}{4}}$, and $F_{m,-1}$ contains $(\Omega + \lambda)S$. Also, if $m, \ell \in \mathbb{N}$ are such that $m \neq \ell$ then $(q^m + 1)q^{\frac{n-2m-1}{4}} \neq \pm(q^\ell + 1)q^{\frac{n-2\ell-1}{4}}$. For details, see [10, Examples 2.8 and 3.12 and Corollary 4.7], where $p = n - 2$.

Proposition 4.6. *For all* $\lambda \in \mathbb{K}$, *the ideal* $(\Omega - \lambda)C_n^q$ *is a completely prime ideal of* C_n^q.

Proof. As $z_{n-1}L_{n-1}^q$ is completely prime in L_{n-1}^q and $z_{n-2} + q\theta(z_{n-2}) \notin z_{n-1}L_{n-1}^q$, this follows on applying the last part of [22, Proposition 1] to $\Omega - \lambda = z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2}) - \lambda$. □

Lemma 4.7. *Let* R *be a right and left noetherian ring with a right and left denominator set* $\mathcal{S}$ *and let* P *be a prime ideal such that* $P \cap \mathcal{S} = \emptyset$. *Then the elements of* $\mathcal{S}$ *are regular modulo* P *and their images in* R/P *form a right and left Ore set with right and left quotient ring* $R_{\mathcal{S}}/PR_{\mathcal{S}}$.

Proof. Let $J = \{r \in R : rs \in P \text{ for some } s \in \mathcal{S}\}$. By a standard argument on the right Ore condition, J is an ideal of R. As R is left noetherian, $J = Rj_1 + \cdots + Rj_n$ for some $j_1, \ldots, j_n \in R$ such that $j_i s_i \in P$ for some $s_1, ..., s_n \in \mathcal{S}$. By [28, Lemma 2.1.8] with $r_1 = \cdots = r_n = 1$, there exists $s \in \mathcal{S}$ such that each $j_i s \in P$ and hence such that $Js \subseteq P$. As J is an ideal containing the prime ideal P and $s \notin P$, it follows that $J = P$. By symmetry, $\{r \in R : sr \in P \text{ for some } s \in \mathcal{S}\} = P$, whence the elements of $\mathcal{S}$ are regular modulo P. For the rest, it is easy to check that $\overline{\mathcal{S}} = \{s + P : s \in \mathcal{S}\}$ is a right and left Ore set in R/P and that $R_{\mathcal{S}}/PR_{\mathcal{S}}$ is a right and left quotient ring of R/P with respect to $\overline{\mathcal{S}}$. □

Theorem 4.8. *Suppose that* q *is not a root of unity. The prime spectrum of* C_n^q *consists of* 0, *the height one prime ideals* $(\Omega - \lambda)C_n^q$, $\lambda \in \mathbb{K}$, *and, for each positive integer* m, *two height two prime ideals* $M_{m,1}$ *and* $M_{m,-1}$ *for which the factors* $C_n^q/M_{m,1}$ *and* $C_n^q/M_{m,-1}$ *have Goldie rank* m.

If $\lambda \neq \pm(q^m + 1)q^{\frac{n-2m-1}{4}}$ *for all* $m \in \mathbb{N}$ *then* $(\Omega - \lambda)C_n^q$ *is maximal. If* $\lambda = (q^m + 1)q^{\frac{n-2m-1}{4}}$ *then* $(\Omega - \lambda)C_n^q \subset M_{m,1}$ *and* $(\Omega + \lambda)C_n^q \subset M_{m,-1}$.

Proof. It follows from Proposition 4.5, the above specification of $\operatorname{Spec} S$ and [28, Proposition 2.1.14] that the prime ideals of C_n^q are as listed. Lemma 4.7 ensures that [28, Lemma 2.2.12] is applicable to show that $S/F_{m,\pm 1}$ and $C_n^q/(F_{m,\pm 1} \cap C_n^q)$ have the same Goldie rank m. □

Corollary 4.9. *Suppose that q is not a root of unity. If $n \geq 3$ then the cyclic connected quantized Weyl algebra C_n^q is a UFD in which all but countably many height one prime ideals are maximal.*

Corollary 4.10. *Suppose that q is not a root of unity. The primitive spectrum of C_n^q consists of the non-zero prime ideals listed in Theorem 4.8.*

Proof. By Proposition 2.18(ii), the ideal 0 of C_n^q is not primitive due to the existence of the central element Ω. All other prime ideals of C_n^q are primitive. For the non-maximal height one prime ideals this is a consequence of Proposition 2.18(iii) and the fact that each of the non-maximal height one prime ideals is strictly contained in a unique height two prime ideal, $F_{m,1}$ or $F_{m,-1}$ for an appropriate value of m. $\square$

5. Automorphisms

In this section we determine the automorphism groups of the algebras L_n^q and C_n^q when $q \neq \pm 1$. We have already observed, in Proposition 2.14, the $\mathbb{K}$-automorphisms ι_ν of L_n^q such that $\iota_\nu(x_i) = \nu^{(-1)^i} x_i$ for $1 \leq i \leq n$, $\nu \in \mathbb{K}^*$, and, when n is odd, the $\mathbb{K}$-automorphisms ι and θ of C_n^q such that $\iota(x_i) = -x_i$ and $\theta(x_i) = x_{i+1}$ for $1 \leq i \leq n$, where indices are interpreted modulo n. Clearly $\theta\iota = \iota\theta$.

Theorem 5.1. *If $n \geq 3$ is odd and $q \neq \pm 1$ then $\mathrm{Aut}_{\mathbb{K}}(C_n^q)$ is cyclic of order $2n$, generated by $\iota\theta$.*

Proof. We interpret the indices in the x_i's modulo n. As $\theta\iota = \iota\theta$, θ has odd order n and ι has order 2, the subgroup $\langle \iota, \theta \rangle$ of $\mathrm{Aut}_K(C_n^q)$ is cyclic of order $2n$, generated by $\iota\theta$.

Let $\psi \in \mathrm{Aut}_{\mathbb{K}}(C_n^q)$. As ψ must send height 1 primes to height 1 primes and $U(C_n^q) = \mathbb{K}$ it follows from Theorem 4.8 that $\psi(\Omega) = \mu(\Omega - \lambda)$ for some $\mu \in \mathbb{K}^*$, $\lambda \in \mathbb{K}$. As Ω has total degree n and degree 1 in each x_i, there must be a permutation $\pi \in S_n$ and scalars $\mu_i \in \mathbb{K}^*$ and $\lambda_i \in \mathbb{K}$ such that, for $1 \leq i \leq n$, $\psi(x_i) = \mu_i x_{\pi(i)} + \lambda_i$.

For $r, s \in C_n^q$, let $[r, s]_q = rs - qrs$. The possibilities for $[x_k, x_\ell]_q$, $1 \leq k, \ell \leq n$, are as follows. If $\ell = k+1$ then $[x_k, x_\ell]_q = 1 - q$, if $\ell = k-1$ then $[x_k, x_\ell]_q = (1-q^2)x_k x_{k-1} + q^2 - q = (q^{-1} - q)x_{k-1}x_k + 1 - q^{-1}$, otherwise, $[x_k, x_\ell]_q \in \mathbb{K}x_k x_\ell = \mathbb{K}x_\ell x_k$. For $1 \leq i \leq n$,

$$\begin{aligned} 1 - q &= \psi([x_i, x_{i+1}]_q) \\ &= [\psi(x_i), \psi(x_{i+1})]_q \\ &= \mu_i \mu_{i+1} [x_{\pi(i)}, x_{\pi(i+1)}]_q + (1-q)(\lambda_i \mu_{i+1} x_{\pi(i+1)} + \lambda_{i+1} \mu_i x_{\pi(i)} + \lambda_i \lambda_{i+1}). \end{aligned}$$

From the possibilities listed above, we see that $\lambda_i = 0$ for all i so $1 - q = \mu_i \mu_{i+1} [x_{\pi(i)}, x_{\pi(i+1)}]_q$. Thus $[x_{\pi(i)}, x_{\pi(i+1)}]_q \in K^*$ but, as $q \neq \pm 1$, the only possibility is that $\pi(i+1) = \pi(i) + 1$. Hence each $\mu_i \mu_{i+1} = 1$ and $\pi = (1\ 2\ 3\ \ldots\ n)^{\pi(1)-1}$. Also $\mu_1^{-1} = \mu_n = \mu_{n-1}^{-1} = \mu_{n-2} = \ldots \mu_2^{-1} = \mu_1$ so $\mu_1 = \pm 1$ and either $\mu_i = 1$ for all i or $\mu_i = -1$ for all i. Thus $\psi = \theta^{\pi(1)-1}$ or $\psi = \iota\theta^{\pi(1)-1}$, whence $\psi \in \langle \iota, \theta \rangle = \langle \iota\theta \rangle$ and therefore $\mathrm{Aut}_K(C_n^q) = \langle \iota\theta \rangle$. $\square$

Theorem 5.2. *If $n \geq 2$ and $q \neq \pm 1$ then the map $\Gamma : \mathbb{K}^* \to \mathrm{Aut}_{\mathbb{K}}(L_n^q)$ given by $\nu \mapsto \iota_\nu$ is a group isomorphism.*

Proof. Certainly Γ is an injective homomorphism. We proceed as in the proof of Theorem 5.1 with z_n, which has total degree n and degree 1 in each x_i, replacing Ω. The height one primes have the form $(z_n-\lambda)L_n^q$, with $\lambda = 0$ if n is odd, so if $\psi \in \mathrm{Aut}_K(L_n^q)$ then $\psi(x_i) = \mu_i x_{\pi(i)}+\lambda_i$ for some permutation $\pi \in S_n$ and some scalars $\mu_i \in \mathbb{K}^*$, $\lambda_i \in \mathbb{K}$. Proceeding as in 5.1, we find that $1-q = \mu_i\mu_{i+1}[x_{\pi(i)}, x_{\pi(i+1)}]_q$ but only for $1 \le i \le n-1$. If $1 \le k \le n-1$ then $[x_n, x_k]_q \notin \mathbb{K}\backslash\{0\}$, as $q \ne \pm 1$, so $\pi(i) \ne n$ for $1 \le i \le n-1$. Hence $\pi(n) = n$ and it follows successively that $\pi(n-1) = n-1, \dots, \pi(1) = 1$, so π is the identity permutation. Although, unlike the cyclic case, it is not necessary that $\mu_n\mu_1 = 1$, we do have $\mu_i\mu_{i+1} = 1$ for $1 \le i \le n-1$ and so $\psi = \iota_\nu$, where $\nu = \mu_1^{-1}$. Thus Γ is an isomorphism. □

6. Quantum cluster algebras

In this section we present two classes of quantum cluster algebras in terms related to the connected quantized Weyl algebras L_n^q and C_n^q when n is odd. We shall not give a full account of the theory of quantum cluster algebras. Helpful references include [3], where the theory was first developed, [20] and, although we will not exploit its multiparameter aspect, [19]. The quantum cluster algebras that we consider here will have no frozen variables. In published definitions of uniparameter quantum cluster algebras, the base ring may be the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ for an indeterminate q, as in [3], or a field which may be either $\mathbb{Q}(q^{\frac{1}{2}})$, with q as before, as in [20], or, more generally, an arbitrary field $\mathbb{K}$ with a distinguished element q that has a square root in $\mathbb{K}$ and is not a root of unity, as in [19]. We shall take the last of these approaches..

Let $n \ge 3$ be odd. Consider the Dynkin quiver Δ_{n-1} of type A_{n-1}, oriented as shown:

$$1 \to 2 \to 3 \to \cdots \to n-2 \to n-1.$$

It is known that all quivers with the Dynkin diagram of type A_{n-1} as underlying graph are mutation equivalent, see [13, Lemma 3.23]. The adjacency matrix is $B = (b_{ij})$ where, for $1 \le i \le j \le n-1$, $b_{ij} = 1$ if $j = i+1$ and $b_{ij} = 0$ if $j \ne i+1$. The inverse of B^T is the skew symmetric $(n-1)\times(n-1)$ integer matrix $\Lambda = (\lambda_{ij})$ where, as in Corollary 3.5, $\lambda_{ij} = 0$ if j is odd or if j and i are both even, and $\lambda_{ij} = 1$ if j is even and i is odd. Hence there is a quantum cluster algebra $\mathcal{A}_{n-1}^q$ on A_{n-1} and, with each $z_jz_i = q^{\lambda_{ij}}z_iz_j$, $(\Delta_{n-1}, \{z_1, z_2, \dots, z_{n-1}\})$ is an initial seed, where $z_1, \dots, z_{n-1}$ are as in Notation 3.6. It will be convenient to amend the quantum cluster $\{z_1, z_2, \dots, z_{n-1}\}$ to $\{y_1, y_2, \dots, y_{n-1}\}$, where $y_i = q^{\frac{1-i}{4}}z_i$ if i is odd and $y_i = q^{\frac{-i}{4}}z_i$ if i is even. By the Laurent phenomenon [3, Corollary 5.2], $\mathcal{A}_n^q$ is a $\mathbb{K}$-subalgebra of $T_{n-1}^q = \mathbb{K}[y_1^{\pm1}, \dots, y_{n-1}^{\pm1}]$. By Proposition 3.7, T_{n-1}^q is simple and $T_n^q = T_{n-1}^q[z_n^{\pm1}]$. The linear connected quantized Weyl algebra L_n^q is a subalgebra of $T_{n-1}^q[z_n]$.

Proposition 6.1. *The quantum cluster algebra $\mathcal{A}_{n-1}^q$ is isomorphic to $L_n^q/(z_n - q^{\frac{n-1}{4}})L_n^q$.*

Proof. Let φ denote the composition of the $\mathbb{K}$-homomorphisms

$$L_n^q \hookrightarrow T_{n-1}^q[z_n] \twoheadrightarrow T_{n-1}^q[z_n]/(z_n - q^{\frac{n-1}{4}})T_{n-1}^q[z_n] \simeq T_{n-1}^q.$$

Note that $\varphi(x_i) = x_i$ for $1 \leq i \leq n-1$ and $\varphi(x_n) = \varphi(z_{n-1}^{-1}(z_{n-2} + z_n)) = z_{n-1}^{-1}(z_{n-2} + q^{\frac{n-1}{4}})$. By [3, Theorem 7.6], $\mathcal{A}_{n-1}^q$ is generated by $y_1, y_2, \ldots, y_{n-1}, w_2, w_3, \ldots, w_n$ where, for $1 < i \leq n$, w_i is the quantum cluster variable obtained by mutation at vertex $i-1$. Note that $y_1 = z_1 = x_1$. By the exchange relations (see, for example [20, 2.2, p702])

$$w_2 = y_1^{-1}(1 + q^{\frac{1}{2}}y_2) = z_1^{-1}(1 + z_2) = x_2 = \varphi(x_2)$$

and, for $3 \leq i \leq n-1$,

$$w_i = \begin{cases} y_{i-1}^{-1}(q^{-\frac{1}{2}}y_{i-2} + y_i) \text{ if } i \text{ is odd} \\ y_{i-1}^{-1}(y_{i-2} + q^{\frac{1}{2}}y_i) \text{ if } i \text{ is even.} \end{cases}$$

In both cases, $w_i = z_{i-1}^{-1}(z_{i-2} + z_i)) = x_i = \varphi(x_i)$. Finally,

$$w_n = y_{n-1}^{-1}(q^{-\frac{1}{2}}y_{n-2} + 1) = z_{n-1}^{-1}(z_{n-2} + q^{\frac{n-1}{2}}) = \varphi(x_n).$$

For $1 < i < n$, y_i is a $\mathbb{K}$-linear combination of $y_{i-1}w_i$ and y_{i-2}, where $y_0 = 1$, so $\mathcal{A}_{n-1}^q$ is generated by $y_1, w_2, w_3, \ldots, w_n$, that is by $\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{n-1})$ and $\varphi(x_n)$. Thus $\mathcal{A}_{n-1}^q = \varphi(L_n^q)$. Clearly $z_n - q^{\frac{n-1}{4}} \in \ker\varphi$ so $\mathcal{A}_{n-1}^q$ is a homomorphic image of $L_n^q/(z_n - q^{\frac{n-1}{4}})L_n^q$. As q is not a root of unity, $L_n^q/(z_n - q^{\frac{n-1}{4}})L_n^q$ is simple, by Theorem 3.13, so $\mathcal{A}_{n-1}^q \simeq L_n^q/(z_n - q^{\frac{n-1}{4}})L_n^q$. □

Corollary 6.2. *The quantum cluster algebra $\mathcal{A}_{n-1}^q$ is simple noetherian.*

Proof. The noetherian condition is immediate from Proposition 6.1 and Proposition 2.18(i) while simplicity is immediate from Proposition 6.1 and Theorem 3.13. □

Remark 6.3. It can be deduced, either from Theorem 6.1 or directly using a similar proof, that L_n^q is the quantum cluster algebra of Δ_n if the vertex n is frozen.

We continue to fix an odd integer $n \geq 3$ and a scalar $q \in K^*$ that is not a root of unity. We now aim to explain the connection between the cyclic connected quantized Weyl algebra $C_n^{q^2}$ and the quantum cluster algebra Q_q of the quiver denoted $P_{n+1}^{(1)}$ in the classification of periodic mutation by Fordy and Marsh [11]. We shall express this quantum cluster algebra as a quotient $U/\Delta U$ of an iterated skew polynomial extension U of $\mathbb{K}$, with Δ central in U. Both U and $U/\Delta U$ contain $C_n^{q^2}$ as a subalgebra.

The quiver $P_{n+1}^{(1)}$ has $n+1$ vertices, which we label 0 to n, and adjacency matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & \ldots & 0 & -1 & 0 \end{pmatrix} = (b_{ij}), \text{ where}$$

$$b_{ij} = \begin{cases} 1 & \text{if } 0 \le i \le n-1 \text{ and } j = i+1 \text{ or } i = 0 \text{ and } j = n, \\ -1 & \text{if } 1 \le i \le n \text{ and } j = i-1 \text{ or } i = n \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The labelling from 0 will also be applied to the rows and columns of the matrices B above and Λ below and has been chosen to fit with the notation $x_1, \dots, x_n$ in a cyclic connected quantized Weyl algebra.

Figure 4 shows $P_6^{(1)}$. In general there are $n+1$ vertices $0, 1, \dots, n$ with a source at 0, a sink at n and a single path from 1 to n. Mutation at the source gives the same diagram but rotated so that the source is at 1 and the sink at 0.

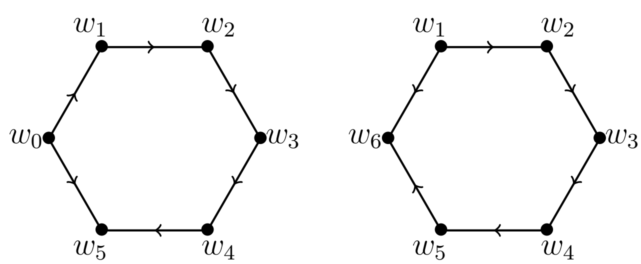

FIGURE 4. $P_6^{(1)}$ before and after mutation at 0

Let $\Lambda = (\lambda_{ij})_{0 \le i,j \le n}$ be the $(n+1) \times (n+1)$ skew-symmetric matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & \dots & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & \dots & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & -1 & 0 & \dots & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & \dots & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}.$$

Thus

$$\lambda_{ij} = \begin{cases} 0 & \text{if } i+j \text{ is even ,} \\ 1 & \text{if } i < j \text{ and } i+j \text{ is odd,} \\ -1 & \text{if } i > j \text{ and } i+j \text{ is odd.} \end{cases}$$

Let T_q be the co-ordinate ring of the quantum $(n+1)$-torus with generators $w_0^{\pm 1}, w_1^{\pm 1}, \dots, w_n^{\pm 1}$ and relations $w_i w_j = q^{\lambda_{ij}} w_j w_i$ for $0 \le i < j \le n$. Thus $w_i w_j = w_j w_i$ if $i+j$ is even and $w_i w_j = q w_j w_i$ if $i < j$ and $i+j$ is odd. Then $B^T \Lambda = 2I_{n+1}$, so, with $\underline{w} := (w_0, w_1, \dots, w_n)$, the triple $(\underline{w}, B, \Lambda)$ is a quantum seed for the quiver $P_{n+1}^{(1)}$ with exchange matrix B, quasi-commutation matrix Λ and initial quantum cluster $\underline{w}$.

Notation 6.4. With Λ and T_q as above, let P_q denote the subalgebra of T_q generated by $w_0, w_1, \dots, w_n$, that is the coordinate ring of $(n+1)$-dimensional quantum affine space with quasi-commutation matrix Λ. Let D_q denote the quotient division algebra of T_q (and P_q). The quantum cluster algebra Q_q for $P^{(1)}_{n+1}$ is the subalgebra of D_q generated by all possible cluster variables and, by the Laurent phenomenon, it is a subalgebra of T_q.

Our next step is to use the exchange relations, [3, (4.23)] or [20, 2.3], to identify some further quantum cluster variables. The mutation illustrated in Figure 4 is in the direction of w_0, where the quiver has its source, and yields the new quantum cluster variable w_{n+1}, and the seed $\{w_1, w_2, \dots, w_{n+1}\}$, where, by the exchange relations,

$$w_{n+1} = w_0^{-1}(1 + qw_1w_n) = (1 + q^{-1}w_1w_n)w_0^{-1}, \tag{15}$$

so that

$$w_0w_{n+1} = 1 + qw_1w_n,\ w_{n+1}w_0 = 1 + q^{-1}w_1w_n \text{ and } w_0w_{n+1} - w_{n+1}w_0 = (q - q^{-1})w_1w_n. \tag{16}$$

Similarly, mutation in the direction of w_n, where the quiver has its sink, gives rise to the quantum cluster variable w_{-1}, and to the seed $\{w_{-1}, w_0, w_1, \dots, w_{n-1}\}$, where

$$w_{-1} = w_n^{-1}(1 + q^{-1}w_0w_{n-1}) = (1 + qw_0w_{n-1})w_n^{-1}.$$

Here the source moves to $n-1$ and the sink to n.

A sequence of mutations where we successively mutate in the direction of the sources $w_0, w_1, \dots$, respectively the sinks $w_n, w_{n-1}, \dots$, will rotate the quiver clockwise, respectively anticlockwise. This gives rise to a countable set $\{w_i\}_{i\in\mathbb{Z}}$ of cluster variables and a countable set of seeds $\{w_i, w_2, \dots, w_{i+n}\}$, $i \in \mathbb{Z}$, with a source at w_i and a sink at w_{i+n}. For $i > n$,

$$w_i = w_{i-n-1}^{-1}(1 + qw_{i-n}w_{i-1}) = (1 + q^{-1}w_{i-n}w_{i-1})w_{i-n-1}^{-1},$$

generalising the formula for w_{n+1}, and, for $i < 0$,

$$w_i = w_{i+n+1}^{-1}(1 + q^{-1}w_{i+1}w_{i+n}) = (1 + qw_{i+1}w_{i+n})w_{i+n+1}^{-1},$$

generalising the formula for w_{-1}. Straightforward calculation shows that, for $i \le j \le i+n$,

$$w_iw_j = \begin{cases} w_jw_i & \text{if } i+j \text{ is even,} \\ qw_jw_i & \text{if } i+j \text{ is odd.} \end{cases} \tag{17}$$

Next we consider mutations in directions of vertices that are neither sources nor sinks. For $i \in \mathbb{Z}$, let $x_i = w_i^{-1}(q^{-\frac{1}{2}}w_{i-1} + q^{\frac{1}{2}}w_{i+1})$. This is the new cluster variable obtained when a seed $\{w_j, w_{j+1}, \dots, w_{j+n}\}$ with $j < i < j+n$ is mutated in the direction of w_i.

Lemma 6.5. *For $i \in \mathbb{Z}$, let w_i and x_1 be as specified above.*

(i) *For all $i \in \mathbb{Z}$ and $k > 0$,*

(a) $x_ix_{i+1} - q^2x_{i+1}x_i = 1 - q^2$,

(b) $x_ix_{i+2k} - q^{-2}x_{i+2k}x_i = 0$,

(c) $x_ix_{i+2k+1} - q^2x_{i+2k+1}x_i = 0$.

(ii) *For* $i \in \mathbb{Z}$,

$$w_i x_i = q^{-\frac{1}{2}} w_{i-1} + q^{\frac{1}{2}} w_{i+1}, \qquad x_i w_i = q^{\frac{1}{2}} w_{i-1} + q^{-\frac{1}{2}} w_{i+1}, \tag{18}$$

$$x_i w_i - q w_i x_i = q^{\frac{1}{2}}(q^{-1} - q) w_{i+1} \text{ and} \tag{19}$$

$$x_i w_i - q^{-1} w_i x_i = q^{\frac{1}{2}}(1 - q^{-2}) w_{i-1}. \tag{20}$$

(iii) *For* $1 \leq i \leq n$ *and* $0 \leq j \leq n$ *with* $j \neq i$,

$$x_i w_j = \begin{cases} q w_j x_i & \text{if } i < j \text{ and } i+j \text{ is even or } i > j \text{ and } i+j \text{ is odd,} \\ q^{-1} w_j x_i & \text{if } i < j \text{ and } i+j \text{ is odd or } i > j \text{ and } i+j \text{ is even.} \end{cases}$$

(iv) *For all* $i \in \mathbb{Z}$, $x_{n+i} = x_i$.

Proof. (i)(a) Using (17),

$$\begin{aligned} x_i x_{i+1} &= (w_i^{-1}(q^{-\frac{1}{2}} w_{i-1} + q^{\frac{1}{2}} w_{i+1}))(w_{i+1}^{-1}(q^{-\frac{1}{2}} w_i + q^{\frac{1}{2}} w_{i+2})) \\ &= w_{i+1}^{-1} w_i^{-1}(w_{i-1} w_i + q w_{i-1} w_{i+2} + w_i w_{i+1} + q^2 w_{i+1} w_{i+2}) \end{aligned}$$

whereas

$$\begin{aligned} x_{i+1} x_i &= (w_{i+1}^{-1}(q^{-\frac{1}{2}} w_i + q^{\frac{1}{2}} w_{i+2}))(w_i^{-1}(q^{-\frac{1}{2}} w_{i-1} + q^{\frac{1}{2}} w_{i+1})) \\ &= w_{i+1}^{-1} w_i^{-1}(q^{-2} w_{i-1} w_i + q^{-1} w_{i-1} w_{i+2} + w_i w_{i+1} + w_{i+1} w_{i+2}) \end{aligned}$$

so $x_i x_{i+1} - q^2 x_{i+1} x_i = 1 - q^2$.

(b)

$$\begin{aligned} x_i x_{i+2k} &= q^{-1} w_i^{-1} w_{i+2k}^{-1}(q^{-\frac{1}{2}} w_{i-1} + q^{\frac{1}{2}} w_{i+1})(q^{-\frac{1}{2}} w_i + q^{\frac{1}{2}} w_{i+2}) \\ &= q^{-1} w_i^{-1} w_{i+2k}^{-1}(q^{-\frac{1}{2}} w_i + q^{\frac{1}{2}} w_{i+2})(q^{-\frac{1}{2}} w_{i-1} + q^{\frac{1}{2}} w_{i+1}) \\ &= q^{-2} w_{i+2k}^{-1}(q^{-\frac{1}{2}} w_i + q^{\frac{1}{2}} w_{i+2}) w_i^{-1}(q^{-\frac{1}{2}} w_{i-1} + q^{\frac{1}{2}} w_{i+1}) \\ &= q^{-2} x_{i+2k} x_i. \end{aligned}$$

A similar calculation establishes (c) but it will be redundant given (iv), for example, as $n-3$ is even, $x_i x_{i+3} = x_{i+n} x_{i+3} = q^2 x_{i+3} x_{i+n} = q^2 x_{i+3} x_i$.

(ii) and (iii) are straightforward from the definition of x_i and (17).

(iv) We show that $x_0 = x_n$. For the general case, add i to all the subscripts.

$$\begin{aligned} x_0 &= w_0^{-1}(q^{-\frac{1}{2}} w_{-1} + q^{\frac{1}{2}} w_1) \\ &= w_0^{-1}(q^{-\frac{1}{2}} w_n^{-1}(1 + q w_0 w_{n-1}) + q^{\frac{1}{2}} w_1) \\ &= q^{\frac{1}{2}} w_n^{-1} w_0^{-1}(1 + q w_0 w_{n-1}) + q^{\frac{1}{2}} w_0^{-1} w_1 \\ &= q^{\frac{1}{2}} w_n^{-1} w_0^{-1} + q^{-\frac{1}{2}} w_n^{-1} w_{n-1} + q^{\frac{1}{2}} w_0^{-1} w_1 \end{aligned}$$

and

$$\begin{aligned} x_n &= w_n^{-1}(q^{-\frac{1}{2}}w_{n-1} + q^{\frac{1}{2}}w_{n+1}) \\ &= w_n^{-1}(q^{-\frac{1}{2}}w_{n-1} + q^{\frac{1}{2}}w_0^{-1}(1+qw_1w_n)) \\ &= q^{-\frac{1}{2}}w_n^{-1}w_{n-1} + q^{\frac{1}{2}}w_n^{-1}w_0^{-1} + q^{\frac{3}{2}}w_n^{-1}w_0^{-1}w_1w_n \\ &= q^{-\frac{1}{2}}w_n^{-1}w_{n-1} + q^{\frac{1}{2}}w_n^{-1}w_0^{-1} + q^{\frac{1}{2}}w_0^{-1}w_1 = x_0. \end{aligned}$$

□

Remark 6.6. By (i)(a)(b) and (ii), the subalgebra C of the quantum cluster algebra Q_q generated by $x_1, x_2, \ldots, x_n$ is a homomorphic image of the cyclic quantized Weyl algebra $C_n^{q^2}$. We shall see later that $C = C_n^{q^2}$.

Theorem 6.7. *The quantum cluster algebra Q_q is generated by $w_0, w_1, x_1, x_2, \ldots x_n$ and, indeed, by $w_1, x_1, x_2, \ldots x_n$.*

Proof. Let S be the subalgebra of Q_q generated by the cluster variables $w_0, w_1, x_1, x_2, \ldots x_n$. As the quiver $P_{n+1}^{(1)}$ is acyclic, it follows from [3, Theorem 7.6] that Q_q is generated by $w_{-1}, w_0, w_1, \ldots, w_n, w_{n+1}, x_1, x_2, \ldots$ and x_n. Although quantum cluster algebras are defined over $\mathbb{Z}[q^{\pm 1/2}]$ in [3], the result in that case implies the result for quantum cluster algebras defined over $\mathbb{K}$. Recall that $w_{-1} = q^{\frac{1}{2}}(w_0x_n - q^{\frac{1}{2}}w_1)$ so $w_{-1} \in S$. Also, for $i \geq 1$, it follows from Lemma 6.5(ii) that $w_{i+1} = q^{\frac{-1}{2}}(w_ix_i - q^{\frac{-1}{2}}w_{i-1})$, from which it follows, inductively, that $w_2, w_3, \ldots, w_n, w_{n+1} \in S$. Therefore $Q_q = S$. Finally, by (20) with $i = 1$,

$$x_1w_1 - q^{-1}w_1x_1 = q^{\frac{1}{2}}(1-q^{-2})w_0$$

so w_0 may be omitted from the list of generators of S. □

Theorem 6.8. (i) *The subalgebra S of T_q generated by $w_0, w_1, x_1, x_2, \ldots, x_{n-1}$ is an iterated skew polynomial extension $\mathbb{K}[w_0][w_1, \rho_0][x_1; \rho_1', \delta_1'] \cdots [x_{n-1}; \rho_{n-1}', \delta_{n-1}']$.*

(ii) *There is a skew polynomial extension $U = S[x'; \beta', \delta']$ of S with a central element Δ such that there is a surjective $\mathbb{K}$-algebra homomorphism $\Gamma : U \to Q_q$ with $\Gamma(x) = x_n$ and $\Gamma(\Delta) = 0$.*

(iii) *ΔU is a completely prime ideal of U.*

(iv) *ΔU is a maximal ideal of U and $Q_q \simeq U/\Delta U$.*

(v) *Q_q is simple and (right and left) noetherian.*

(vi) *Q_q is generated by $w_0, w_1, x_1, x_2, \dots, x_n$ subject to the relations*

$$\begin{aligned}
w_0w_1 &= qw_1w_0 \\
x_jw_0 &= q^{(-1)^{j+1}}w_0x_j, && \textit{if } 1 \le j < n, \\
x_jw_1 &= q^{(-1)^j}w_1x_j, && \textit{if } 1 < j \le n, \\
x_1w_1 &= q^{-1}w_1x_1 + q^{\frac{1}{2}}(1-q^2)w_0, \\
x_nw_0 &= qw_0x_n + q^{\frac{-1}{2}}(1-q^2)w_1, \\
x_ix_{i+1} &= q^2x_{i+1}x_i + 1 - q^2, && \textit{if } 1 \leqslant i \leqslant n-1, \\
x_nx_1 &= q^2x_1x_n + 1 - q^2, \\
x_ix_j &= q^2x_jx_i, && \textit{if } i \geqslant 1, i+1 < j \leqslant n \textit{ and } j-i \textit{ is odd}, \\
x_ix_j &= q^{-2}x_jx_i, && \textit{if } i \geqslant 1, i+1 < j < n \textit{ and } j-i \textit{ is even}, \\
w_0w_{n+1} &= qw_1w_n + 1.
\end{aligned}$$

Here w_n and w_{n+1} are defined recursively using the formula $w_j = q^{-\frac{1}{2}}w_{j-1}x_{j-1} + q^{-1}w_{j-2}$, $j \ge 2$, and are linear combinations of standard monomials of the form $w_1^a w_0^b x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$, where $a = 1$ and $b = 0$ or $a = 0$ and $b = 1$ and each $d_i \le 1$.

Proof. For $0 \le i \le n$, let $T_q^{(i)}$ be the subalgebra of T_q generated by $w_0^{\pm 1}, w_1^{\pm 1}, \dots, w_i^{\pm 1}$, let R_i be the subalgebra generated by $w_0, w_1^{\pm 1}, \dots, w_i^{\pm 1}$ and, for $i > 1$, let S_i be the subalgebra generated by $w_0, w_1, x_1, x_2, \dots x_{i-1}$. Thus, as $x_{i-1} = w_{i-1}^{-1}(q^{-\frac{1}{2}}w_{i-2} + q^{\frac{1}{2}}w_i)$, we have $S_i \subset R_i \subset T_q^{(i)}$ for $i > 1$. Also $T_q^{(n)} = T_q$ and, for $0 \le i \le n-1$, $T_q^{(i+1)}$ and R_{i+1} are, respectively, the skew Laurent polynomial rings $T_q^i[w_i^{\pm 1}; \rho_i]$ and $R_i[w_i^{\pm 1}; \rho_i]$, where the automorphism ρ_i of T_q^i or R_i, as appropriate, is such that, for $1 \le j \le i$, $\rho_i(w_j) = w_j$ if $i+j$ is even and $\rho_i(w_j) = q^{-1}w_j$ if $i+j$ is odd. Here, and elsewhere in the proof, we abuse notation in using the same notation for ρ_i and its restriction to a subalgebra.

(i) Let $1 \le i \le n-1$. In the skew polynomial ring $R_i[w_{i+1}; \rho_i]$, x_i has degree one and invertible leading coefficient. It follows from Lemma 3.8 that $R_i[w_{i+1}; \rho_i] = R_i[x_i; \rho_i', \delta_i']$ for an appropriate automorphism ρ_i' of R_i and ρ_i'-derivation δ_i'. By Lemma 6.5(i), for $1 \le j \le i-1$, $\rho_i'(x_j) = q^{\pm 2}x_j$ and $\delta_i'(x_j) = 0$ or $1 - q^{-2}$. By Lemma 6.5(ii) and (iii), $\rho_i'(w_0) = q^{(-1)^{i+1}}w_0$, $\delta_i'(w_0) = 0$, $\rho_i'(w_1) = q^{(-1)^i}w_1$ and $\delta_i'(w_1) = 0$ unless $i = 1$ in which case, by (20), $\delta_1'(w_1) = q^{\frac{1}{2}}(1 - q^{-2})w_0$. Thus $\rho_i'(S_i) = S_i$ and $\delta_i'(S_i) \subseteq S_i$ for $x \in \{w_0, w_1, x_1, \dots, x_{i-1}\}$ and it follows inductively that, for $1 \le i \le n$,

$$S_i = \mathbb{K}[w_0][w_1, \rho_0][x_1; \rho_1', \delta_1'] \cdots [x_{i-1}; \rho_{i-1}', \delta_{i-1}'].$$

In particular, this holds for $i = n$ where $S_{n-1} = S$.

(ii) Let A be the quantum n-torus generated by $w_1^{\pm 1}, \dots, w_n^{\pm 1}$ and let α be the $\mathbb{K}$-automorphism of A such that $\alpha(w_i) = w_i$ if i is even and $\alpha(w_i) = qw_i$ if i is odd. The $\mathbb{K}$-subalgebra R_n of T_n^q has the form $A[w_0; \alpha]$ while the $\mathbb{K}$-subalgebra generated by $w_1^{\pm 1}, \dots, w_n^{\pm 1}$ and w_{n+1} is $A[w_{n+1}; \alpha^{-1}]$. Observe that w_1w_n is central in A and that $\alpha(w_1w_n) = q^2w_1w_n$.

Let $u = 1 + q^{-1}w_1w_n$ so that $v := u - \alpha(u) = (q^{-1} - q)w_1w_n$. Form the conformal ambiskew polynomial ring $R = R(A, \alpha, v, 1)$, with w_0 in the role of y. Thus $R = A[w_0; \alpha][x; \beta, \delta]$ where $\beta(w_0) = w_0$, $\delta(w_0) = (q^{-1} - q)w_1w_n$ and, for $1 \leq i \leq n$, $\beta(w_i) = \alpha^{-1}(w_i)$ and $\delta(w_i) = 0$. Note that $x_i \in A$ if $1 < i \leq n-1$ and $x_1 = w_1^{-1}(q^{\frac{-1}{2}}w_0 + q^{\frac{1}{2}}w_2) \in A[w_0; \alpha]$. The Casimir element $\Delta := xw_0 - q^{-1}w_1w_n - 1 = xw_0 - q^{-1}w_nw_1 - 1$ is central in R.

As $w_{n+1}w_j = \beta(w_j)w_{n+1}$ for $1 \leq j \leq n$, and, by (16), $w_{n+1}w_0 - w_0w_{n+1} = (q-q^{-1})w_1w_n - 1$, there is a $\mathbb{K}$-algebra homomorphism $\Psi : R \to T_q$ such that $\Psi(w_i) = w_i$ for $0 \leq i \leq n$, $\Psi(x) = w_{n+1}$ and $\Psi(\Delta) = 0$. Note that $\Psi(x_i) = x_i$ for $1 \leq i \leq n-1$.

Let $x' = w_n^{-1}(q^{\frac{1}{2}}x + q^{\frac{-1}{2}}w_{n-1})$, so that $\Psi(x') = x_n$. By Lemma 3.8, $R = A[w_0; \alpha][x'; \beta', \delta']$ for an appropriate $\mathbb{K}$-automorphism β' of $A[w_0; \alpha]$ and an appropriate β'-derivation δ' of $A[w_0; \alpha]$.

The next step is to show that $\beta'(S) = S$ and $\delta'(S) \subseteq S$ so that β' and δ' restrict, respectively, to an automorphism and β'-derivation δ' of S, which we also denote by β' and δ', giving rise to a subalgebra U of R of the form $S[x'; \beta', \delta']$.

Let $0 \leq i \leq n$. Using the formulae in Lemma 3.8, we see that $\beta'(w_i) = q^{(-1)^i}w_i$ and

$$\delta'(w_i) = q^{\frac{1}{2}}w_n^{-1}\delta(w_i) + q^{-\frac{1}{2}}w_n^{-1}w_{n-1}w_i - q^{-\frac{1}{2}}\beta'(w_i)w_n^{-1}w_{n-1}.$$

As $\delta(w_i) = 0$ if $i > 0$ and $\delta(w_0) = (q^{-1} - q)w_1w_n$, it follows that $\delta'(w_0) = q^{\frac{1}{2}}(q^{-1} - q)w_1$, $\delta'(w_i) = 0$ for $1 \leq i \leq n-1$ and $\delta'(w_n) = q^{\frac{1}{2}}(1 - q^{-2})w_{n-1}$.

Now let $1 \leq i \leq n-1$ and recall that $x_i = w_i^{-1}(q^{-\frac{1}{2}}w_{i-1} + q^{\frac{1}{2}})w_{i+1}$. It follows that $\beta'(x_i) = q^{2(-1)^{i-1}}x_i$. Also $\delta'(x_i) = 0$ for $2 \leq i \leq n-2$ whereas

$$\delta'(x_1) = q^{-\frac{1}{2}}\beta'(w_1^{-1})\delta'(w_0) = 1 - q^2$$

and

$$\delta'(x_{n-1}) = q^{\frac{1}{2}}\beta'(w_{n-1}^{-1})\delta'(w_n) = 1 - q^{-2}.$$

The above calculations establish that for each generator $g \in \{w_0, w_1, x_1, \ldots, x_{n-1}\}$ of S, g is an eigenvector of β', with non-zero eigenvalue, and $\delta'(g) \in S$. Hence there is an iterated skew polynomial extension $U = S[x'; \beta', \delta'] \subseteq R$ as described above. As U is generated by $w_0, w_1, x_1, \ldots, x_{n-1}$ and x', the image $\Psi(U)$ is generated by $w_0, w_1, x_1, \ldots, x_{n-1}$ and x_n. By Theorem 6.7, $\Psi(S) = Q_q$. Note that $x = q^{-\frac{1}{2}}w_nx' - q^{-1}w_{n-1} \in U$ so $\Delta \in U$. As Δ is in central in R and $\Delta \in U$, Δ is central in U. So (ii) holds with Γ being the restriction to U of Ψ.

(iii) As $x = q^{-\frac{1}{2}}(w_nx' - q^{\frac{-1}{2}}w_{n-1})$ and $\Delta = xw_0 - q^{-1}w_1w_n - 1 = w_0x - qw_1w_n - 1$, $\Delta = q^{-\frac{1}{2}}w_0w_nx' + d$ where $d = -q^{-1}w_0w_{n-1} - qw_1$. Both w_0 and w_n are normal in S, with S/w_0S and S/w_nS isomorphic to skew polynomial rings over domains, so w_0S and w_nS are completely prime ideals. Note that $d \notin w_0S$ and $d \notin w_nS$ by easy degree arguments. If $r, s \in S$ are such that $rd = w_0w_ns$ then $r = w_0e$ for some $e \in S$, $ed \in w_nS$ and $e \in w_nS$, whence $r \in w_nw_0S$. Thus d is regular modulo w_0w_nS. By [22, Proposition 1], ΔU is a completely prime ideal of U.

(iv) Note that $R/\Delta R$ is the generalized Weyl algebra $W(A, \alpha, u)$. We shall apply [24, Theorem 5.4] to see that $W(A, \alpha, u)$ is simple. Criteria (i) and (ii) of that theorem hold because the skew Laurent polynomial ring $A[x^{\pm 1};\alpha]$ is simple, by [27, Proposition 1.3], whence A is α-simple and α^m is outer for all $m \geq 1$. Criterion (iii), the regularity of u, is clear and (iv) holds because, for $m \geq 1$, $uA + \alpha^m(u)A$ contains the unit $(1-q^{2m})w_1w_n$. By [24, Theorem 5.4], $W(A,\alpha,u)$ is simple.

Let P be the subalgebra of A generated by $w_1,\dots,w_n$. The automorphism α of A restricts to an automorphism of P, which we also denote α, and v is central in P so we can form the ambiskew polynomial ring $B := R(P,\alpha,v,1)$. Thus $B \subset U \subset R$. The multiplicatively closed set $\mathcal{S}$ generated by $\mathbb{K}^* \cup \{w_i : 1 \leq i \leq n\}$ is a right and left Ore set of regular elements in P, with ring of quotients A, and it follows from [14, Lemma 1.4], that $\mathcal{S}$ is a right and left Ore set of regular elements in B, with ring of quotients R. As $B \subset U \subset R$, R is also the ring of quotients of U with respect to $\mathcal{S}$. Let Q be the completely prime ideal ΔU of U. By degree, $w_i \notin Q$ for $1 \leq i \leq n$ so $\Delta R \cap U = \Delta U$, by [18, Theorem 10.20] or [28, Proposition 2.1.16].

Let J be a proper ideal of U strictly containing ΔU. By [28, Proposition 2.1.16], JR is an ideal of R containing ΔR. If $JR = \Delta R$ then $J \subseteq \Delta R \cap U = \Delta U$ so $\Delta R \subset JR$. By the simplicity of $R/\Delta R$, it follows that $JR = R$ and hence, by [28, Proposition 2.1.16(vi,iv)] J contains a monomial $w_1^{m_1}w_2^{m_2}\dots w_n^{m_n}$ for some non-negative integers $m_1, m_2, \dots, m_n$ with at least one m_i non-zero. Define the *weight* of a monomial $w_1^{m_1}w_2^{m_2}\dots w_n^{m_n}$ to be $\sum_{i=1}^n im_i$ and let $w = w_1^{m_1}w_2^{m_2}\dots w_n^{m_n}$ be a monomial of least weight in J. Let i, $1 \leq i \leq n$, be maximal such that $m_i \neq 0$. Suppose that $i = n$. In the skew polynomial ring U, $\beta'(w_n) = q^{-1}(w_n)$ and $\delta'(w_n) = \mu w_{n-1}$, where $\mu = q^{\frac{1}{2}}(1-q^{-2}) \in \mathbb{K}^*$. We claim that, for $k \geq 1$, $\delta'(w_n^k) = [k]_{q^{-2}}\mu w_{n-1}w_n^{k-1}$. This holds for $k = 1$ and if it holds for $k = m-1$ then

$$\begin{aligned}\delta'(w_n^m) &= \beta'(w_n^{m-1})\delta'(w_n) + \delta'(w_n^{m-1})w_n \\ &= q^{1-m}w_n^{m-1}\mu w_{n-1} + [m-1]_{q^{-2}}\mu w_{n-1}{w_n}^{m-2}w_n \\ &= \mu([m-1]_{q^{-2}} + q^{-2(m-1)})w_{n-1}{w_n}^{m-1} \\ &= \mu[m]_{q^{-2}}w_{n-1}{w_n}^{m-1}.\end{aligned}$$

By induction, it holds for all m. With w as above and $w' = w_1^{m_1}w_2^{m_2}\dots w_{n-1}^{m_{n-1}}$, so that $w = w'w_n^{m_n}$, $\beta'(w') = q^cw'$ for some $c \in \mathbb{Z}$ and $\delta'(w') = 0$. Hence $\delta'(w) = q^cw'\mu[m_n]_{q^{-2}}w_{n-1}w_n^{m_n-1}$. Now $\delta'(w) = x'w - \beta'(w)x' \in J$ as $\beta'(w) = q^{c-m_n}w$. But $q^c\mu[m_n]_{q^{-2}} \neq 0$ and $w'w_{n-1}w_n^{m_n-1}$ has weight one lower than that of w so, by minimality of w, $m_n = 0$. Repeating the argument with U, x', β', δ' replaced by $S_i, x_i, \rho_i', \delta_i'$, $i = n-1,\dots,2,1$, for which, by (20), $\rho_i'(w_i) = q^{-1}(w_i)$ and $\delta_i'(w_i) = \mu w_{i-1}$, we see that $m_{n-1} = \dots = m_2 = 0$, so that $w = w_1^{m_1}$, and that $w_0w_1^{m_1-1} \in J$. Recall that $\Delta = xw_0 - q^{-1}w_nw_1 - 1 \in J$ from which it follows that $xw_0w_1^{m_1-1} - q^{-1}w_nw_1^m - w_1^{m_1-1} \in J$ and hence that $w_1^{m_1-1} \in J$, contradicting the minimality of w. This contradiction shows that ΔU is a maximal ideal of U. But $\Delta U \subseteq \ker\Gamma$ so $\Delta U = \ker\Gamma$ and $U/\Delta U \simeq \Gamma(U) = Q_q$.

(v) The simplicity of Q_q is immediate from (iv) and the noetherian conditions follow from (ii), (iv) and Hilbert's Basis Theorem for skew polynomial rings [18, Theorem 2.6].

(vi) Follows from (ii), (iv) and [9, Proposition 1].

$\square$

Corollary 6.9. *There is a $\mathbb{K}$-automorphism θ of Q_q such that, for $i \in \mathbb{Z}$, $\theta(x_i) = x_{i+1}$, $\theta^{-1}(x_i) = x_{i-1}$, $\theta(w_i) = w_{i+1}$ and $\theta^{-1}(w_i) = w_{i-1}$.*

Proof. This follows easily from Theorem 6.8(vi) and Lemma 6.5. □

Remark 6.10. The automorphism θ in Corollary 6.9 lifts to a $\mathbb{K}$-automorphism Θ of the iterated skew polynomial ring U in Theorem 6.8(ii) with $\Theta(x_i) = x_{i+1}$ for $1 \leq i \leq n-1$, $\Theta(x') = x_1$, $\Theta(w_0) = w_1$, $\Theta(w_1) = q^{-\frac{1}{2}}(w_1x_1 - q^{-\frac{1}{2}}w_0)$ and $\Theta(\Delta) = \Delta$. We leave the proof to the interested reader.

Remark 6.11. The cyclic connected quantized Weyl algebra $C_n^{q^2}$ embeds in U in an obvious way with $x_n \mapsto x'$. As $\Omega - \lambda$ cannot, by degree, be in ΔU, it follows from Theorem 6.8 that this induces an embedding $C_n^{q^2} \hookrightarrow Q_q$. Also $C_n^{q^2}$ is a homomorphic image of U, being isomorphic to U/I, where $I = w_0U + w_1U$, which, by Theorem 6.8(vi), is an ideal of U.

Remark 6.12. It will be shown in the PhD thesis of first author that the set of quantum cluster variables in Q_q is the union of n θ-orbits, namely the infinite orbit $\{w_i : i \in \mathbb{Z}\}$ and the $n-1$ finite θ-orbits $\{\theta^j(z_k) : 0 \leq j \leq n-1\}$, $1 \leq k \leq n-1\}$, where the elements z_k are as in Section 5.

7. Poisson structures

The connected quantized Weyl algebras L_n^q and C_n^q are quantizations, in the sense of [4, III.5.4], of Poisson algebras. In other words there are Poisson brackets on the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$ that are semiclassical limits of the families L_n^q and C_n^q. The Poisson algebra that is the semiclassical limit of $C_n^{q^2}$ was introduced by Fordy [12] and this sparked our interest in C_n^q. In this section we shall present results of an analysis of the Poisson prime spectrum of the semiclassical limits of L_n^q and C_n^q. This analysis was carried out in parallel with that of the prime spectra of L_n^q and C_n^q and the methods in the two mirror each other. Good references for Poisson algebras, Poisson ideals, Poisson prime ideals, Poisson cores and the Poisson centre include [17] and [17].

In the remainder of the paper, we assume, as before, that the field $\mathbb{K}$ is algebraically closed but also that it has characteristic 0. Let $n \in \mathbb{N}$ and let F_n, H_n denote, respectively, the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$ and the Laurent polynomial algebra $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition 7.1. Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ skew-symmetric matrix over $\mathbb{K}$. On each of F_n and H_n, there is a Poisson bracket, the *log-canonical Poisson bracket*, such that, for $1 \leq i, j \leq n$,

$$\{x_i, x_j\} = \lambda_{ij}x_ix_j.$$

Note that, for $m_1, \ldots, m_n \in \mathbb{Z}$,

$$\{x_i, x_1^{m_1} \ldots x_n^{m_n}\} = (m_1\lambda_{i1} + \cdots + m_n\lambda_{in})x_1^{m_1} \ldots x_i^{m_i+1} \ldots x_n^{m_n}. \tag{21}$$

The simplicity criterion for quantum tori given by [27, Proposition 1.3] has the following Poisson analogue, where $\mathrm{PZ}(H_n)$ denotes the Poisson centre of H_n.

Proposition 7.2. *Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ skew-symmetric matrix over $\mathbb{K}$. Then, for the log-canonical Poisson bracket determined by Λ on the Laurent polynomial algebra $H_n = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the following are equivalent.*

(i) *If $m_1, \dots, m_n \in \mathbb{Z}$ are such that $m_1\lambda_{i1} + \cdots + m_n\lambda_{in} = 0$ for $1 \leq i \leq n$ then $m_i = 0$ for all i.*
(ii) $\mathrm{PZ}(H_n) = \mathbb{K}$.
(iii) *H_n is Poisson simple.*

Proof. This follows from [30, Lemma 1.2], the proof of which, although presented over $\mathbb{C}$, is valid for any base field of characteristic 0. □

Lemma 7.3. *Let S be a simple Poisson algebra over $\mathbb{K}$ with Poisson centre $\mathbb{K}$ and extend the Poisson bracket to the polynomial algebra $S[t]$ with $\{t, s\} = 0$ for all $s \in S$. Then the non-zero Poisson prime ideals of $S[t]$ are the ideals of the form $(t - \lambda)S[t]$ for some $\lambda \in \mathbb{K}$.*

Proof. As $\operatorname{char} \mathbb{K} = 0$ it follows from [15, Lemma 6.2] that the Poisson core I of any maximal ideal of S is prime. By Poisson simplicity, $I = 0$ so S is an integral domain. It is clear that the ideals $(t - \lambda)S[t]$ are Poisson prime, with $S[t]/(t - \lambda)S[t] \simeq S$. Let P be a non-zero proper Poisson prime ideal of $S[t]$ and let d be the minimal degree in t of non-zero elements of P. Then $d > 0$ as $P \cap S$ is a Poisson ideal of S and must be 0. It is easy to verify that

$$J := \{s \in S : st^d + s_{d-1}t^{d-1} + \cdots + s_0 \in P \text{ for some } s_{d-1}, \dots, s_0 \in S\}$$

is a Poisson ideal in S. By Poisson simplicity, $1 \in J$ so there exist $s_{d-1}, \dots, s_0 \in S$ such that

$$f := t^d + s_{d-1}t^{d-1} + \cdots + s_0 \in P.$$

For each $s \in S$, $\deg(\{s, f\}) < d$ so $s_{d-1}, \dots, s_0 \in \mathrm{PZ}(S) = \mathbb{K}$. Thus the prime ideal $P \cap \mathbb{K}[t]$ is non-zero and, as $\mathbb{K}$ is algebraically closed, $(t - \lambda)S[t] \subseteq P$ for some $\lambda \in \mathbb{K}$. By the Poisson simplicity of $S[t]/(t - \lambda)S[t]$, $P = (t - \lambda)S[t]$. □

In Remark 2.2, we observed that the semiclassical limit of the relation $xy - qyx = 1 - q$ is $\{x, y\} = xy - 1$. A similar discussion shows that the semiclassical limits of the relations $xy - qyx = 0$ and $xy - q^{-1}yx = 0$ are given by $\{x, y\} = xy$ and $\{x, y\} = -xy$ respectively. The semiclassical limit of L_n^q is the polynomial algebra F_n with the Poisson bracket given by

$$\begin{aligned}
\{x_i, x_{i+1}\} &= x_i x_{i+1} - 1, && \text{if } 1 \leqslant i \leqslant n-1, \\
\{x_i, x_j\} &= x_i x_j, && \text{if } i \geqslant 1, i+1 < j \leqslant n \text{ and } j - i \text{ is odd}, \\
\{x_i, x_j\} &= -x_i x_j, && \text{if } i \geqslant 1, i+1 < j \leqslant n \text{ and } j - i \text{ is even}.
\end{aligned}$$

This can be made formal by applying the quantization procedure described in [15, 2.1] and [4, III.5.4] to the algebra obtained from L_n^q on replacing the parameter q by a central invertible indeterminate Q and taking $h = Q - 1$. The Poisson algebra obtained on equipping F_n with this bracket will be denoted F_n^L.

Similarly, for odd $n \geq 3$, the family C_n^q, $q \in \mathbb{K}^*$, has semiclassical limit F_n with Poisson bracket given by

$$\begin{aligned}
\{x_i, x_{i+1}\} &= x_i x_{i+1} - 1, && \text{if } 1 \leqslant i \leqslant n-1,\\
\{x_i, x_j\} &= x_i x_j, && \text{if } i \geqslant 1, i+1 < j \leqslant n, \text{ and } j-i \text{ is odd},\\
\{x_i, x_j\} &= -x_i x_j, && \text{if } i \geqslant 1, i+1 < j < n, \text{ and } j-i \text{ is even},\\
\{x_n, x_1\} &= x_n x_1 - 1.
\end{aligned}$$

The Poisson algebra obtained on equipping F_n with this bracket will be denoted F_n^C. There is a Poisson automorphism θ of F_n^C, analogous to the automorphism θ of C_n^q in 2.14(iv), given by $\theta(x_i) = x_{i+1}$, where subscripts are taken modulo n in $\{1, 2, \ldots, n\}$.

Notation 7.4. We specify n distinguished elements $z_1, z_2, \ldots, z_n$ of F_n^L by the same recurrence formula as in Section 3: $z_{-1} = 0, z_0 = 1$ and, for $i \geq 0$, $z_{i+1} = z_i x_{i+1} - z_{i-1}$. Note that, if T_L is the localization of F_n at the multiplicatively closed set generated by $z_1, z_2, \ldots, z_n$ then $T_L = \mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}]$ because, for $1 \leq j \leq n$, $x_j = (z_j + z_{j-2})z_{j-1}^{-1}$. It follows that $z_1, z_2, \ldots, z_n$ are algebraically independent. Let $S_L = \mathbb{K}[z_1, z_2, \ldots, z_n]$ and $U = \mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]$ and note that $U[z_n]$ is the localization of F_n at the multiplicatively closed set generated by $z_1, z_2, \ldots, z_{n-1}$.

The formulae listed in the following result can be deduced from Lemmas 3.2, 3.3 and 3.4 and Corollary 3.5 by passing to the semiclassical limit or by direct calculation.

Lemma 7.5. *In the Poisson algebra F_n^L, the following hold.*

(i) *For $1 \leq i \leq n$, $z_i = x_1\theta(z_{i-1}) - \theta^2(z_{i-2})$.*

(ii) *For $1 \leq i, j \leq n$,*

$$\{x_i, z_j\} = \begin{cases} (-1)^{i+1} x_i z_j & \text{if } j \text{ is odd and } j < i-1,\\ 0 & \text{if } j \text{ is even and } j < i-1,\\ z_{i-2} - z_{i-1}x_i = -z_i & \text{if } j \text{ is odd and } j = i-1,\\ z_{i-2} & \text{if } j \text{ is even and } j = i-1,\\ 0 & \text{if } j \text{ is odd and } j \geq i,\\ (-1)^{i-1} x_i z_j & \text{if } j \text{ is even and } j \geq i. \end{cases}$$

(iii) *For $1 \leq i < j \leq n$,*

$$\{z_i, z_j\} = \begin{cases} 0 & \text{if } j \text{ is odd or } i, j \text{ are both even},\\ z_i z_j & \text{if } j \text{ is even and } i \text{ is odd}. \end{cases}$$

Thus S_L is a Poisson subalgebra of F_n^L with the log-canonical Poisson bracket determined by the $n \times n$ skew symmetric matrix $\Lambda_n = (\lambda_{ij})$ such that, for $j > i$,

$$\lambda_{ij} = \begin{cases} 1 & \text{if } j \text{ is even and } i \text{ is odd},\\ 0 & \text{otherwise}. \end{cases}$$

Lemma 7.6. *If n is odd then the ideals $(z_n-\lambda)F_n$, $\lambda\in\mathbb{K}$, are Poisson prime ideals of F_n^L. If n is even then z_nF_n and the ideals $z_nF_n+(z_{n-1}-\lambda)F_n$, $\lambda\in\mathbb{K}^*$ are Poisson prime ideals of F_n^L.*

Proof. It follows easily from Lemma 7.5(ii) that the listed ideals are all Poisson.

As $z_n-\lambda=z_{n-1}x_n-z_{n-2}-\lambda$ and, by degree, $z_{n-2}-\lambda\notin z_{n-1}F_{n-1}$, it is readily checked, by induction, that $z_n-\lambda$ is irreducible in F_n. It follows, as F_n is a UFD, that $(z_n-\lambda)F_n$ is prime for all λ. Thus $(z_n-\lambda)F_n$ is Poisson prime for all λ if n is odd and when $\lambda=0$ if n is even.

Suppose that n is even and that $\lambda\neq 0$. As $x_n\equiv\lambda^{-1}z_{n-2}\bmod (z_nF_n+(z_{n-1}-\lambda)F_n)$, there is a Poisson isomorphism between $F_n/(z_nF_n+(z_{n-1}-\lambda)F_n)$ and $F_{n-1}/(z_{n-1}-\lambda)F_{n-1}$ so $z_nF_n+(z_{n-1}-\lambda)F_n$ is Poisson prime. □

Proposition 7.7. *For $n\geq 2$, let S_L, T_L and U be as in Notation 7.4.*
(i) *If n is even then T_L is Poisson simple.*
(ii) *If n is odd then $T_L=U[z_n^{\pm1}]$, U is a Poisson simple subalgebra of T_L and z_n is Poisson central.*
(iii) *The non-zero proper Poisson prime ideals of F_n^L are the ideals $(z_n-\lambda)F_n$, $\lambda\in\mathbb{K}$, if n is odd, and z_nF_n and the ideals $z_nF_n+(z_{n-1}-\lambda)F_n$, $\lambda\in\mathbb{K}^*$, if n is even.*

Proof. (i) This follows from Lemma 7.5(iii) and Proposition 7.2. In the application of the latter, which is analogous to the proof of Lemma 3.7, the rows of Λ_n should be considered in the order $2,4,6,\dots,n,n-1,n-3,n-5,\dots,1$.

(ii) If n is odd, z_n is Poisson central by Lemma 7.5(iii) and U is Poisson simple by the even case.

(iii) By Lemma 7.6, the listed ideals are Poisson prime. Let P be a non-zero Poisson prime ideal of F_n. By Lemma 7.5(ii), if $z_m\in P$ for some $m<n$ then $z_j\in P$ for $0\leq j\leq m$ and, in particular $1=z_0\in P$. So $z_m\notin P$ for $m<n$ and $PU[z_n]$ is a Poisson prime ideal of $U[z_n]$, which, as observed in 7.4, is the localization of F_n at the multiplicatively closed set generated by $z_1,z_2,\dots,z_{n-1}$.

Suppose that n is odd. By (ii), U is Poisson simple so, by Lemma 7.3, $PU[z_n]=(z_n-\lambda)U[z_n]$ for some $\lambda\in\mathbb{K}$. As $(z_n-\lambda)F_n$ is prime in F_n it follows from standard localization theory, for example [29, Theorem 5.32], that $P=(z_n-\lambda)F_n$.

Now suppose that n is even. By (ii), T_L, which is the localization of $U[z_n]$ at the powers of z_n, is Poisson simple so $z_n\in PU[z_n]$. If $PU[z_n]=z_nU[z_n]$ then $P=z_nF_n$ by [29, Theorem 5.32], so we can assume that $PU[z_n]\supset z_nU[z_n]$. As $U[z_n]/z_nU[z_n]\simeq U$, it follows from the odd case that $PU[z_n]=z_nU[z_n]+(z_{n-1}-\lambda)U[z_n]$ for some $\lambda\in\mathbb{K}^*$. By [29, Theorem 5.32], $P=z_nF_n+(z_{n-1}-\lambda)F_n$. □

We now turn our attention to the Poisson algebra F_n^C. Note that, for $1\leq i<n$, the Poisson subalgebra $\mathbb{K}[x_1,\dots,x_i]$ coincides with F_i^L. The Poisson brackets among the elements $z_1,z_2,\dots,z_{n-1}$ and $x_1,x_2,\dots,x_{n-1}$ are as before. The following Lemma can be deduced from Lemma 4.1 by passing to the semiclassical limit or by direct calculation.

Lemma 7.8. *Let $n\geq 3$ be odd and let $\Omega=z_{n-1}x_n-z_{n-2}-\theta(z_{n-2})\in F_n$.*

(i) *For* $1 \leq j \leq n-2$,

$$\{x_n, z_j\} = \begin{cases} z_j x_n - \theta(z_{j-1}) & \text{if } j \text{ is odd,} \\ -\theta(z_{j-1}) & \text{if } j \text{ is even.} \end{cases}$$

(ii) $\{x_n, z_{n-1}\} = z_{n-2} - \theta(z_{n-2})$.
(iii) $\theta(\Omega) = \Omega$.
(iv) Ω *is Poisson central in* F_n^C.

One might expect that, by analogy with F_n^L when n is odd, the non-zero Poisson prime ideals of F_n^C would be the ideals $(\Omega - \lambda)F_n$. However there are two exceptional non-zero Poisson primes M_λ, $\lambda = \pm 1$, such that $F_n^C/M_\lambda \simeq F_{n-2}^L/(z_{n-2} - \lambda)F_{n-2}^L$. To establish the existence of these, we shall need to calculate $\{z_{n-3}, \theta(z_{n-3})\}$.

Lemma 7.9. *Let* $n \geq 3$ *be odd. The following hold in* F_n^C.

(i) $\{x_1, \theta(z_{n-3})\} = -\theta^2(z_{n-4}) = -x_1\theta(z_{n-3}) + z_{n-2}$.
(ii) *Let* $2 \leq i \leq n-2$. *Then* $\{x_i, \theta(z_{n-3})\} = (-1)^i x_i \theta(z_{n-3})$.
(iii) *Let* $0 \leq i \leq n-3$. *Then*

$$\{z_i, \theta(z_{n-3})\} = \begin{cases} -\theta^{i+1}(z_{n-i-3}) & \text{if } i \text{ is odd,} \\ z_i\theta(z_{n-3}) - \theta^{i+1}(z_{n-i-3}) & \text{if } i \text{ is even.} \end{cases}$$

(iv) $\{z_{n-3}, \theta(z_{n-3})\} = z_{n-3}\theta(z_{n-3}) - 1$.

Proof. (i) By Lemma 7.8(i), $\{x_n, z_{n-3}\} = -\theta(z_{n-4})$. The result follows by applying θ and using Lemma 3.2.

(ii) By Lemma 7.5(ii), $\{x_{i-1}, z_{n-3}\} = (-1)^i x_{i-1} z_{n-3}$ and the result again follows on applying θ.

(iii) The result is true when $i = 0$, in which case $z_i = 1$, and, by (i), when $i = 1$, in which case $z_i = x_1$. Let $i \geq 1$ and suppose that the result holds for i and for $i-1$. If i is even then

$$\begin{aligned} &\{z_{i+1}, \theta(z_{n-3})\} \\ =&\{z_i x_{i+1} - z_{i-1}, \theta(z_{n-3})\} \\ =&- z_i x_{i+1}\theta(z_{n-3}) + z_i x_{i+1}\theta(z_{n-3}) - \theta^{i+1}(z_{n-i-3})x_{i+1} + \theta^i(z_{n-i-2}) && \text{(by (ii) and induction)} \\ =&- \theta^i(\theta(z_{n-i-3})x_1 - z_{n-i-2}) \\ =&- \theta^i(\theta^2(z_{n-i-4})) && \text{(by 7.5(i))} \\ =&- \theta^{i+2}(z_{n-i-4}), \end{aligned}$$

which is the result for $i+1$ in this case. If i is odd then

$$\begin{aligned}
&\{z_{i+1}, \theta(z_{n-3})\}\\
=&\{z_i x_{i+1} - z_{i-1}, \theta(z_{n-3})\}\\
=&-x_{i+1}\theta^{i+1}(z_{n-i-3}) + z_i x_{i+1}\theta(z_{n-3}) - z_{i-1}\theta(z_{n-3}) + \theta^i(z_{n-i-2}) \text{ (by (ii) and induction)}\\
=&\theta^i(-x_1\theta(z_{n-i-3}) + z_{n-i-2}) + \theta(z_{n-3})(z_i x_{i+1} - z_{i-1})\\
=&-\theta^i(\theta^2(z_{n-i-4})) + \theta(z_{n-3})z_{i+1} \qquad \text{(by 7.5(i))}\\
=&-\theta^{i+2}(z_{n-i-4}) + \theta(z_{n-3})z_{i+1},
\end{aligned}$$

which is again the result for $i+1$ in this case. The result follows by induction.

(iv) This is the special case of (iii) when $i = n-3$. □

Lemma 7.10. (i) *Let $\lambda = \pm 1$, let $\tau_\lambda : F_n \to F_{n-2}$ be the $\mathbb{K}$-algebra homomorphism such that $\tau_\lambda(x_i) = x_i$ for $1 \le i \le n-2$, $\tau_\lambda(x_{n-1}) = \lambda z_{n-3}$ and $\tau_\lambda(x_n) = \lambda\theta(z_{n-3})$, let $\pi_\lambda : F_{n-2} \to F_{n-2}/(z_{n-2}-\lambda)F_{n-2}$ be the canonical epimorphism and let $\rho_\lambda = \pi_\lambda \circ \tau_\lambda$. Then $\rho_\lambda : F_n^C \to F_{n-2}^L/(z_{n-2}-\lambda)F_{n-2}^L$ is a Poisson homomorphism.*

(ii) *For $\lambda = \pm 1$, let $M_\lambda = \ker \rho_\lambda$. Then M_λ is a Poisson prime ideal of F_n^C and is maximal as a Poisson ideal of F_n^C. As an ideal of F_n, M_λ is generated by $z_{n-2}-\lambda$, $x_{n-1} - \lambda z_{n-3}$ and $x_n - \lambda\theta z_{n-3}$. Also $z_{n-1} \in M_\lambda$, $\theta(z_{n-2}) - \lambda \in M_\lambda$ and $\Omega + 2\lambda \in M_\lambda$.*

Proof. (i) Write τ, π and ρ for τ_λ, π_λ and ρ_λ respectively. We need to show that $\rho(\{x_i, x_j\}) = \{\rho(x_i), \rho(x_j)\}$ for $1 \le i < j \le n$. This is clear when $j \le n-2$. Let $j = n-1$. If $i \le n-3$ then $\tau(\{x_i, x_{n-1}\}) = (-1)^{i-1}\lambda x_i z_{n-3}$ and, by Lemma 7.5(ii),

$$\{\tau(x_i), \tau(x_{n-1})\} = \lambda\{x_i, z_{n-3}\} = (-1)^{i-1}\lambda x_i z_{n-3}.$$

It follows immediately that $\rho(\{x_i, x_j\}) = \{\rho(x_i), \rho(x_j)\}$.

Also

$$\tau(\{x_{n-2}, x_{n-1}\}) = \tau(x_{n-2}x_{n-1} - 1) = \lambda x_{n-2}z_{n-3} - 1 = \lambda(z_{n-2} + z_{n-4}) - 1$$

whereas, by Lemma 7.5(ii)

$$\{\tau(x_{n-2}), \tau(x_{n-1})\} = \lambda\{x_{n-2}, z_{n-3}\} = \lambda z_{n-4}.$$

As $\pi(\lambda(z_{n-2} + z_{n-4}) - 1 - \lambda z_{n-4} = \lambda^2 - 1 = 0$, it follows, in this case also, that $\rho(\{x_i, x_j\}) = \{\rho(x_i), \rho(x_j)\}$.

Now let $j = n$. If $2 \le i \le n-2$ then a calculation similar to that in the case $j = n-1$, $i \le n-3$, but with Lemma 7.9 rather than Lemma 7.5, shows that $\tau(\{x_i, x_j\}) = \{\tau(x_i), \tau(x_j)\} = (-1)^i\lambda x_j\theta(z_{n-3})$ and hence that $\rho(\{x_i, x_j\}) = \{\rho(x_i), \rho(x_j)\}$. This leaves the cases $i = 1$ and $i = n-1$. In the latter, $\tau(\{x_{n-1}, x_n\}) = \tau(x_{n-1}x_n - 1) = \lambda^2 z_{n-3}\theta(z_{n-3}) - 1 = z_{n-3}\theta(z_{n-3}) - 1$, as $\lambda^2 = 1$, and, by Lemma 7.9(iv), $\{\tau(x_{n-1}), \tau(x_n)\} = \lambda^2\{z_{n-3}, \theta(z_{n-3})\} = z_{n-3}\theta(z_{n-3}) - 1$. It follows that $\rho(\{x_{n-1}, x_n\}) = \{\rho(x_{n-1}), \rho(x_n)\}$.

Finally, $\tau(\{x_1, x_n\}) = \tau(1 - x_1x_n) = 1 - \lambda x_1\theta(z_{n-3}) = 1 - \lambda(z_{n-2} + \theta^2(z_{n-4}))$, by 7.5(i), whereas, by Lemma 7.8(i),

$$\{\tau(x_1), \tau(x_n)\} = \lambda\{x_1, \theta(z_{n-3})\} = \lambda\theta(\{x_n, z_{n-3}\}) = -\lambda\theta^2(z_{n-4}).$$

As $\pi(1-\lambda(z_{n-2}+\theta^2(z_{n-4}))+\lambda\theta^2(z_{n-4}))=1-\lambda^2=0$, it follows again that $\rho(\{x_i,x_j\})=\{\rho(x_i),\rho(x_j)\}$.

(ii) It is clear that ρ_λ is surjective and hence, by Lemma 7.7 and the First Isomorphism Theorem for Poisson algebras, that M_λ is Poisson prime and maximal as a Poisson ideal. Clearly $z_{n-2}-\lambda\in M_\lambda$, $x_{n-1}-\lambda z_{n-3}\in M_\lambda$ and $x_n-\lambda\theta z_{n-3}\in M_\lambda$. Also, $M_\lambda\cap F_{n-2}$ is Poisson prime in F^L_{n-2} and, by Lemma 7.7, must be $(z_{n-2}-\lambda)F_{n-2}$. Let $w=x_{n-1}-\lambda z_{n-3}$ and $y=x_n-\lambda\theta z_{n-3}$. Then $F_n=F_{n-2}[w,y]$ and so $M_\lambda=M_\lambda\cap F_{n-2}+wF_{n-2}+yF_{n-2}=(z_{n-2}-\lambda)F_{n-2}+wF_{n-2}+yF_{n-2}$. Also

$$z_{n-1}=z_{n-2}x_{n-1}-z_{n-3}=(z_{n-2}-\lambda)x_{n-1}+\lambda x_{n-1}-z_{n-3}\in M_\lambda,$$
$$\theta(z_{n-2})-\lambda=z_{n-2}-\lambda-\{x_n,z_{n-1}\}\in M_\lambda,$$

by Lemma 7.8(ii), and

$$\Omega+2\lambda=z_{n-1}x_n-(z_{n-2}-\lambda)-(\theta(z_{n-2}-\lambda)\in M_\lambda.$$

□

Proposition 7.11. *Let $n\geq 3$ be odd and let S_C be the polynomial algebra $\mathbb{K}[z_1,\dots,z_{n-1},\Omega]$ and T_C be the Laurent polynomial algebra $\mathbb{K}[z_1^{\pm1},\dots,z_{n-1}^{\pm1},\Omega^{\pm1}]$.*
(i) *S_C is a Poisson subalgebra of F_n^C and the Poisson brackets on S_C and T_C are the log-canonical Poisson brackets determined by Λ_n.*
(ii) *$T_C=U[\Omega^{\pm1}]$ where $U=\mathbb{K}[z_1^{\pm1},z_2^{\pm1},\dots,z_{n-1}^{\pm1}]$ is a Poisson simple subalgebra of T_C and Ω is Poisson central.*
(iii) *The non-zero proper Poisson prime ideals of F_n^C are the ideals $(\Omega-\lambda)F_n$, $\lambda\in\mathbb{K}$, and the two ideals M_1 and M_{-1} from Lemma 7.10.*
(iv) *For $\mu\in\mathbb{K}$, the Poisson algebra $F_n^C/(\Omega-\mu)F_n^C$ is Poisson simple if and only if $\mu\neq\pm2$.*

Proof. The proofs of (i) and (ii) are completely analogous to those of the odd part of Proposition 7.7, with Ω replacing z_n.

(iii) The ideals $M_{\pm1}$ are Poisson prime by Lemma 7.10. The ideal $(\Omega-\lambda)F_n^C$ is Poisson, as Ω is Poisson central by Lemma 7.8(iv) and prime, as for $(z_n-\lambda)F_n$ in the odd part of Proposition 7.7. So all the ideals listed are Poisson prime.

Let P be a non-zero Poisson prime ideal of F_n^C. By Lemma 7.5(ii), if $z_m\in P$ for some $m<n-1$ then $z_j\in P$ for $0\leq j\leq m$ and, in particular $1=z_0\in P$. So $z_m\notin P$ for $m<n-1$. If also $z_{n-1}\notin P$ then $P=(\Omega-\lambda)F_n^C$ for some $\lambda\in\mathbb{K}$ as in the proof of the odd part of Proposition 7.7(iii), with Ω replacing z_n. So we may assume that $z_{n-1}\in P$.

By Lemma 7.8(ii), $z_{n-2}-\theta(z_{n-2})=\{x_n,z_{n-1}\}\in P$. So $z_{n-2}-\theta(z_{n-2})\in P\cap F_{n-1}$. By total degree, $z_{n-2}-\theta(z_{n-2})\notin z_{n-1}F_{n-1}$ so, by Proposition 7.7(iii), $P\cap F_{n-1}$, which is Poisson prime in F^L_{n-1}, must have the form $z_{n-1}F_n+(z_{n-2}-\mu)F_{n-1}$ for some $\mu\in\mathbb{K}^*$. As $z_{n-2}-\theta(z_{n-2})\in P$, we also have that $\theta(z_{n-2})-\mu\in P$.

Let $\lambda=\mu^{-1}$. Note that $z_{n-2}x_{n-1}-z_{n-3}=z_{n-1}\in P$ so $x_{n-1}-\lambda z_{n-3}\in P$. Also $\{x_n,\lambda z_{n-2}-1\}\in P$ so, by Proposition 7.8(i), $\lambda(z_{n-2}x_n-\theta(z_{n-3}))\in P$. Hence $x_n-\lambda\theta(z_{n-3})\in P$. Therefore $\{x_{n-1},x_n-\lambda\theta(z_{n-3})\}\in P$ and, as $x_{n-1}\equiv\lambda z_{n-3}\bmod P$,

$$\{x_{n-1},x_n\}-\lambda^2\{z_{n-3},\theta(z_{n-3})\}\in P.$$

Using Lemma 7.9(iv),

$$x_{n-1}x_n - 1 - \lambda^2(z_{n-3}\theta(z_{n-3}) - 1) \in P. \tag{$*$}$$

But we also have that $x_{n-1}(x_n - \lambda\theta(z_{n-3})) \in P$ so $x_{n-1}x_n - \lambda x_{n-1}\theta(z_{n-3}) \in P$ and $x_{n-1}x_n - \lambda^2 z_{n-3}\theta(z_{n-3}) \in P$. Combining this with (*), $\lambda^2 - 1 \in P$ so we must have $\lambda = \pm 1$ and $\lambda = \mu$. We now know that P contains $z - \lambda$, $x_{n-1} - \lambda z_{n-3}$ and $x_n - \lambda\theta(z_{n-3})$ and it then follows from Lemma 7.10 that $M_\lambda \subseteq P$. By the maximality of M_λ, $P = M_\lambda$.

(iv) This is immediate from (iii) and the fact that, by Lemma 7.10(ii), $\Omega + 2\lambda \in M_\lambda$. □

8. Commutative cluster algebras with Poisson structure

Let $n \geq 3$ be odd. Let $n \geq 3$ be odd. In this section we aim to present the commutative cluster algebras $\mathcal{A}$ of the quivers A_{n-1} and $P^{(1)}_{n+1}$ considered in Section 6 as Poisson simple algebras $J/\Delta J$, where J is a polynomial algebra with a Poisson bracket and Δ is a Poisson central element of J.

We first consider $P^{(1)}_{n+1}$. If $w_0, w_1, \ldots, w_n$ are the initial cluster variables then $\mathcal{A}$ is a Poisson subalgebra of the Laurent polynomial algebra $R := \mathbb{K}[w_0^{\pm 1}, w_1^{\pm 1}, \ldots, w_n^{\pm 1}]$ with the log-canonical bracket such that, for $0 \leq i, j \leq n$, $\{w_i, w_j\} = \lambda_{ij}w_iw_j$ where, if $j \geq i$,

$$\lambda_{ij} = -\lambda_{ji} = \begin{cases} 1 & \text{if } j - i \text{ is odd,} \\ 0 & \text{if } j - i \text{ is even.} \end{cases} \tag{22}$$

With the matrices B and Λ as in Section 6, [13, Theorem 1.4] ensures that the above Poisson bracket $\{-,-\}$ is compatible with the cluster algebra $\mathcal{A}$, in other words, for each seed $\{y_1, y_2, \ldots, y_n\}$ and for $0 \leq i, j \leq n$, $\{y_i, y_j\} = \lambda'_{ij}y_iy_j$ for some antisymmetric $(n+1) \times (n+1)$ matrix $\Lambda' = (\lambda'_{ij})$.

Lemma 8.1. *The Poisson algebra R is Poisson simple.*

Proof. Let $m_0, m_1, \ldots m_n \in \mathbb{Z}$ be such that $m_0\lambda_{i0} + \cdots + m_n\lambda_{in} = 0$ for $0 \leq i \leq n$. From the cases $i = 0$ and $i = n-1$,

$$m_1 + m_3 + \cdots + m_{n-2} + m_n = 0 = -m_1 - m_3 \cdots - m_{n-2} + m_n,$$

whence $m_n = 0$. Similarly the cases $i = 1$ and $i = n$ give

$$-m_0 + m_2 + \cdots + m_{n-1} = 0 = -m_0 - m_2 + \cdots - m_{n-1},$$

and $m_0 = 0$. Repeating the argument, deleting the first and last columns at each stage, gives $0 = m_{n-1} = m_1 = m_{n-2} = m_2 = \cdots = m_{(n+1)/2}$. By Proposition 7.2, R is Poisson simple. □

As in the quantum case, there are new cluster variables w_i and x_i, $i \in \mathbb{Z}$, such that, for $i > n$,

$$w_i = w_{i-n-1}^{-1}(1 + w_{i-n}w_{i-1})$$

for $i < 0$,

$$w_i = w_{i+n+1}^{-1}(1 + w_{i+n}w_{i+1}),$$

and, for $i \in \mathbb{Z}$,

$$x_i = w_i^{-1}(w_{i-1} + w_{i+1}).$$

The following can be deduced from the quantum counterpart, Lemma 6.5, by taking the semiclassical limit or directly from (22). Parts (i) and (iv) were observed by Fordy in [12].

Lemma 8.2. *With $x_1, x_2, \dots, x_n$ as specified above,*

(i) *For all $i \in \mathbb{Z}$ and $k > 0$,*
   (a) $\{x_i, x_{i+1}\} = 2(x_i x_{i+1} - 1)$,
   (b) $\{x_i, x_{i+2k}\} = -2x_{i+2k}x_i$,
   (c) $\{x_i, x_{i+2k+1}\} = 2x_{i+2k+1}x_i$.

(ii) *For $i \in \mathbb{Z}$,*
$$w_i x_i = w_{i-1} + w_{i+1} \tag{23}$$
and
$$\{x_i, w_i\} = x_i w_i - 2w_{i+1} = 2w_{i+1} - x_i w_i \tag{24}$$

(iii) *For $1 \leq i \leq n$ and $0 \leq j \leq n$ with $j \neq i$,*
$$\{x_i, w_j\} = \begin{cases} x_i w_j & \text{if } i < j \text{ and } i+j \text{ is even or } i > j \text{ and } i+j \text{ is odd,} \\ -x_i w_j & \text{if } i < j \text{ and } i+j \text{ is odd or } i > j \text{ and } i+j \text{ is even.} \end{cases}$$

(iv) *For all $i \in \mathbb{Z}$, $x_{n+i} = x_i$.*

Lemma 8.3. *For $i \leq j \leq i+n$,*
$$\{w_i, w_j\} = \begin{cases} 0 & \text{if } i+j \text{ is even,} \\ w_i w_j & \text{if } i+j \text{ is odd} \end{cases} \tag{25}$$
and
$$\{w_0, w_{n+1}\} = 2w_1 w_n. \tag{26}$$

Proof. These are straightforward calculations and are omitted. □

By [2, Corollary 1.21], the cluster algebra $\mathcal{A}$ is the subalgebra of R generated by the cluster variables $w_{-1}, w_0, w_1, \dots w_n, w_{n+1}, x_1, x_2, \dots x_n$. By Lemma 8.2(iv), $x_0 = x_n$ so, by (23), $w_{-1} = w_0 x_n - w_1$ and, for $j \geq 2$, $w_j = w_{j-1}x_{j-1} - w_{j-2}$. Hence the list of generators can be reduced to $w_0, w_1, x_1, x_2, \dots x_n$. By Lemmas 8.2(iv) and 8.3, $\mathcal{A}$ is a Poisson subalgebra of R.

Notation 8.4. Let $D = \mathbb{K}[W_0, W_1, \dots, W_{n+1}]$ be a polynomial algebra in $n+2$ variables and let $E = \mathbb{K}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_n^{\pm 1}, W_{n+1}]$. Let J be the subalgebra of E generated by $W_0, W_1, X_1, X_2, \dots, X_n$, where, for $1 \leq i \leq n$, $X_i = W_i^{-1}(W_{i-1} + W_{i+1})$. Observe that $W_{i+1} = W_i X_i - W_{i-1}$ from which it follows that $D \subseteq J \subseteq E$ and hence that J is a polynomial algebra in $n+2$ indeterminates $W_0, W_1, X_1, X_2, \dots, X_n$.

Lemma 8.5. *There is a Poisson bracket on D, and hence on E, such that, for $0 \le i < j \le n+1$,*

$$\{W_i, W_j\} = \begin{cases} W_iW_j & \text{if } j-i \text{ is odd,} \\ 0 & \text{if } j-i \text{ is even and } j-i < n+1, \\ 2W_1W_n & \text{if } i=0 \text{ and } j=n+1. \end{cases}$$

Also J is a Poisson subalgebra of D.

Proof. Note that the given rules for $\{W_i, W_m\}$ determine log-canonical Poisson brackets on each of $\mathbb{K}[W_1, \dots, W_{n+1}]$ and $\mathbb{K}[W_0, W_1, \dots, W_n]$. To establish the Jacobi identity on D, it suffices to check it on the triples (W_0, W_i, W_{n+1}), $0 < i < n+1$. If i is odd then $\{\{W_0, W_i\}, W_{n+1}\} = W_0W_iW_{n+1} + 2W_1W_iW_n$, $\{\{W_i, W_{n+1}\}, W_0\} = -W_0W_iW_{n+1} - 2W_1W_iW_n$ and $\{\{W_{n+1}, W_0\}, W_i\} = -2\{W_1W_n, W_i\} = 0$ so the Jacobi identity holds. If i is even then $\{\{W_0, W_i\}, W_{n+1}\} = 0 = \{\{W_i, W_{n+1}\}, W_0\}$ and $\{\{W_{n+1}, W_0\}, W_i\} = -2\{W_1W_n, W_i\} = 0$ so the Jacobi identity holds again. Also, for $1 \le i \le n$, $\{W_i, -\}$ is a derivation on D and $\{W_{n+1}, -\}$ is the restriction to D of the derivation $\sum_{i=0}^{n+1} f_i\partial_i$, where each $\partial_i = \frac{\partial}{\partial W_i}$, $f_0 = -2W_1W_n$, $f_i = 0$ if $i > 0$ is even and $f_i = -W_1W_{n+1}$ if i is odd. As $\{r+s, -\} = \{r, -\} + \{s, -\}$ and $\{rs, -\} = r\{s, -\} + s\{r, -\}$ for, $\{p, -\}$ is a derivation for all $p \in D$. So we have a Poisson bracket on D and hence, by [25, Lemma 1.3], on E.

The following can be deduced from the Poisson bracket on D.

$$\{X_i, X_{i+1}\} = 2(X_iX_{i+1} - 1) \quad \text{if } i < n, \tag{27}$$
$$\{X_n, X_1\} = 2(X_nX_1 - 1), \tag{28}$$
$$\{X_i, X_{i+2k}\} = -2X_{i+2k}X_i \quad \text{if } 1 \le i \le i+2k \le n, \tag{29}$$
$$\{X_i, X_{i+2k+1}\} = 2X_{i+2k+1}X_i \quad \text{if } 1 \le i \le i+2k+1 \le n, \tag{30}$$
$$\{X_i, W_i\} = W_{i-1} - W_{i+1} \tag{31}$$
$$= X_iW_i - 2W_{i+1} \tag{32}$$
$$= 2W_{i-1} - X_iW_i, \tag{33}$$
$$\{X_n, W_0\} = X_nW_0 - 2W_1, \tag{34}$$
$$\{X_i, W_j\} = X_iW_j \quad \text{if } i+j \text{ is odd and } j=0 \text{ or } j=1, \tag{35}$$
$$\{X_i, W_j\} = -X_iW_j \quad \text{if } i+j \text{ is even, } j \ne i \text{ and } j=0 \text{ or } j=1. \tag{36}$$

It follows that J is a Poisson subalgebra of D.

□

Proposition 8.6. *Let $\Delta = W_0W_{n+1} - W_1W_n - 1$. Then Δ is Poisson central in D, E and J. Also $\Delta D, \Delta E$ and ΔJ are Poisson prime ideals of D, E and J respectively and both $E/\Delta E$ and $J/\Delta J$ are Poisson simple.*

Proof. For $1 \le i \le n$ either i is even and $\{W_i, W_{n+1}\} = 0 = \{W_i, W_0\}$ and $\{W_i, W_n\}W_1 = -\{W_i, W_1\}W_n$ or i is odd and $\{W_i, W_{n+1}\}W_0 = -\{W_i, W_0\}W_{n+1}$ and $\{W_i, W_n\} = 0 =$

$\{W_i, W_1\}$. In both cases, $\{W_i, \Delta\} = 0$. Also

$$\begin{aligned}\{W_{n+1}, \Delta\} &= W_{n+1}(-2W_nW_1) + 2W_{n+1}W_nW_1 = 0 \text{ and}\\ \{W_0, \Delta\} &= W_0(2W_nW_1) - 2W_0W_nW_1 = 0.\end{aligned}$$

Hence $\Delta \in \mathrm{PZ}(D)$. It follows that $\Delta \in \mathrm{PZ}(E)$ and $\Delta \in \mathrm{PZ}(J)$.

As Δ is irreducible and Poisson central in D, ΔD is a Poisson prime ideal of D. Observe that E is the localization of D, and also of the intermediate ring J, at the multiplicatively closed set $\mathcal{W}$ generated by $W_0, W_1, \dots, W_n$ and that $\mathcal{W} \cap \Delta D = \emptyset$. It follows that ΔE is a Poisson prime ideal of E so that Δ is irreducible in E. As $W_0, W_1, \dots, W_n$ are, by an easy induction using the formula $W_{i+1} = W_iX_i - W_{i-1}$, prime elements of J, it follows that Δ is irreducible in the UFD J. Hence ΔJ is a Poisson prime ideal of J.

There is a Poisson algebra homomorphism ϕ from R to $E/\Delta E$ given by $w_i \mapsto W_i + \Delta E$ for $0 \le i \le n$. As $W_{n-1} + \Delta E = \phi(w_0^{-1}w_1w_n)$, ϕ is surjective and as R is Poisson simple, by Lemma 8.1, ϕ is an isomorphism. Thus $E/\Delta E$ is Poisson simple.

Suppose that $J/\Delta J$ is not Poisson simple. By [23, 3.3(ii)], J has a Poisson primitive, and hence Poisson prime, ideal Q such that $\Delta J \subset Q \subset J$. Recall that E is the localization of J at $\mathcal{W}$ so QE is a Poisson prime ideal of E strictly containing ΔE. By the Poisson simplicity of $E/\Delta E$, $QE = E$ and therefore $Q \cap \mathcal{W} \neq \emptyset$. As Q is prime, $W_i \in Q$ for some i with $0 \le i \le n$. By (34), $\{X_n, W_0\} = X_nW_0 - 2W_1$ and if $i > 0$ then, by (32) and (33), $\{X_i, W_i\} = X_iW_i - 2W_{i+1} = 2W_{i-1} - X_iW_i$. It follows that $W_j \in Q$ for $0 \le j \le n+1$. But $W_0W_{n+1} - W_1W_n - 1 = \Delta \in Q$ so $1 \in Q$, contradicting the fact that Q is proper. Thus $J/\Delta J$ is Poisson simple. □

Proposition 8.7. *The cluster algebra $\mathcal{A}$ is isomorphic to the simple Poisson algebra $J/\Delta J$.*

Proof. Recall the Poisson isomorphism ϕ from R to $E/\Delta E$ such that $w_i \mapsto W_i + \Delta E$ and note that $x_i \mapsto X_i + \Delta E$. We have observed that $\mathcal{A}$ is the subalgebra of R generated by $w_0, w_1, x_1, x_2, \dots x_n$ and is a Poisson subalgebra of R. Hence $\mathcal{A} \simeq \phi(\mathcal{A})$ which is generated by $W_0 + \Delta E$, $W_1 + \Delta E$ and the n elements $X_i + \Delta E$, $1 \le i \le n$. As $J \cap \Delta E = \Delta J$ by standard localization theory, $J/\Delta J$ embeds in $E/\Delta E$ by $b + \Delta J \mapsto b + \Delta E$ and $J/\Delta J$ is generated by $W_0 + \Delta J$, $W_1 + \Delta J$ and the n elements $X_i + \Delta J$, $1 \le i \le n$ so $J/\Delta J \simeq \phi(A)$. Thus $\mathcal{A} \simeq J/\Delta J$. □

Remarks 8.8. In accordance with the quantum case, $\mathcal{A}$ has a Poisson automorphism θ such that $\theta(w_i) = w_{i+1}$, giving rise to an infinite orbit $\{w_i\}_{i \in \mathbb{Z}}$, and $\theta(x_i) = x_{i+1}$, giving rise to a finite orbit $\{w_i\}_{1 \le i \le n}$. The full set of cluster variables in $\mathcal{A}$ is the union of n θ-orbits, namely the infinite orbit $\{w_i : i \in \mathbb{Z}\}$ and the $n-1$ finite θ-orbits $\{\theta^j(z_k) : 0 \le j \le n-1\}$, $1 \le k \le n-1\}$, where the elements z_k are as in Section 7.

The Poisson analogue of Theorem 6.7, where w_0 could be omitted from the generators, is more subtle. Here $w_0 = (\{x_1, w_1\} + x_1w_1)/2$ so the omission of w_0 from the generators would require an appropriate definition of generators of a Poisson algebra.

In the case of the Dynkin quivers A_{n-1} we have the following result. The proof is omitted, as is that of the subsequent analogue of Corollary 6.2. It is parallel to that of Proposition 6.1.

The change of initial cluster variables from z_i to y_i is redundant and Poisson simplicity of $F^L/(z_n-1)F^L$, from Proposition 7.7(iii), replaces simplicity of $L_n^q/(z_n-q^{(2-n)/2})L_n^q$.

Proposition 8.9. *Let $n \geq 3$ be odd and let F^L be the polynomial algebra $\mathbb{K}[x_1,\dots,x_n]$ equipped with the Poisson bracket corresponding to L_n^q. The cluster algebra $\mathcal{A}_{n-1}$ of the Dynkin quiver of type A_{n-1} is isomorphic, as a Poisson algebra, to $F^L/(z_n-1)F^L$.*

Corollary 8.10. *Let $n \geq 3$ be odd. The cluster algebra $\mathcal{A}_{n-1}$ is a simple Poisson algebra.*

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