Diffusion algebras.

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PhD Thesis.

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September 2005.

Contents

Chapter 0

Introduction

0.0 Notation

Throughout this thesis, I shall use the following notational conventions:

- $\mathbb{C}$ denotes the field of complex numbers.
- $\mathbb{R}$ denotes the field of real numbers.
- $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geqslant 0\}$ denotes the set of positive reals, a real number $x > 0$ shall be referred to as *strictly* positive (likewise negative and strictly negative).
- $\mathbb{Z}$ denotes the ring of integers.
- $\mathbb{Q}$ denotes the field of rational numbers.
- $\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the set of natural numbers.
- $\mathbb{Z}^+ := \{0, 1, 2, \dots\}$ denotes the set of positive integers.
- We will often use diagrams such as:

$$\begin{array}{ccccc} \circ & \cdots & \circ & \cdots & \circ \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{array}$$

  to depict prime spectra. Solid lines denote inclusion where the higher ideal contains the lower. Horizontal dotted lines joining two families of primes indicates that there is a nontrivial intersection of the two families.

0.1 Abstract

In this thesis we shall provide a ring theoretic investigation of diffusion algebras.

Diffusion algebras were first defined in [IPR01] by Isaev, Pyatov and Rittenberg (see also [PT02]). Their representations are thought to be interesting in the study of a broad class of stochastic processes. These are linear processes involving finitely many sites each populated by one of a finite number of different species of particle. In an infinitesimal time step the (unnormalised, i.e. the probability of the certain event, $\mathbb{P}(\Sigma)$, is allowed to be something other than one in the law of conservation of probablity) probability of two neighbouring particles is given and depends on which species of particle the two are and on which way around they are. For example, a process with two species of particle (empty and occupied) and with the probability of them swapping one way around being zero gives the *fully asymmetric exclusion process* which can be used to model blood flow through narrowed capilleries. It is thought that the probability distribution of such a process can be studied by obtaining matrices satisfying relations involving the probabilities mentioned above and certain auxiliary matrix–valued parameters. Under the *ansatz* that these auxiliary parameters can be chosen to be scalar matrices, the defining relations of a diffusion algebra arise.

We will first obtain a simple criterion to establish whether or not a diffusion algebra is noetherian. The classification of these two subclasses of diffusion algebras will then be handled seperately.

In the noetherian case, we will show that a diffusion algebra in arbitrarily many generators is isomorphic to one of three types of algebra. The first, the multiparameter quantum affine n–space is a well studied object (see [GL98]). The second is quite closely related to the quantised Weyl algebra, in the sense that it contains the 2–generator quantised Weyl algebra as a subalgebra and any other generators q–commute with these two generators in the most general way that is compatible with the defining relation of the 2–generator quantised weyl algebra. The third case is closely related to the universal enveloping algebra of the 2–d soluble Lie algebra, in that it is again built from copies of that algebra with all the other relations given by q–commutation in the most general way compatible with these relations. The defining relations of diffusion algebras very much

resemble deformations of universal enveloping algebras of certain Lie algebras so it is not much of a surprise to see this algebra playing a role.

We will analyse the prime spectrum of the noetherian diffusion algebras, including the overwhelming majority of (although not all) root–of–unity cases. We will show how the prime spectra can be used to distinguish the three types of noetherian diffusion algebra and we will identify which of these ideals are primitive. Generically, we will discover that all the primes except for a unique height $n-1$ one are primitive for an n–generator noetherian diffusion algebra. Lastly, we will show how the prime and primitive spectra of diffusion algebras have a stratified structure (in the sense of [BG02]).

In the non–noetherian case we will classify all 3–generator diffusion algebras. We will find that there is a rather large number of different algebras such that each 3–generator non–noetherian diffusion algebra is either isomorphic or anti–isomorphic to one of them. Nevertheless, we shall proceed to classify the prime spectrum of each case. We shall use the prime spectra, along with some other tricks, to distinguish the various types up to isomorphism. We shall identify which of these prime ideals are primitive. We shall also present an example of a 4–generator diffusion algebra which exhibits a form of mixed behaviour which suggests that there is no nice answer to the classification of non–noetherian diffusion algebras in arbitrarily many generators in the way that there was for noetherian ones.

0.2 Acknowledgements

I would like to thank my PhD supervisor, David Jordan, for his help and advice; Kate, my family and friends for their support; Mark Young and the maths staff at Redhill comprehensive and Mike Keating and the maths staff at Imperial college for interesting me in maths and specifically non–commmutative algebra in the first place.

Chapter 1

Preliminaries

1.1 Rings and ideals

Here I will recall the definition of the basic objects of study for this thesis.

1.1.1 Definitions

- A ring is an additive abelian group R, such that $R \setminus \{0\}$ is a multiplicative semigroup where the multiplication and addition are related by distributivity:

$$(r + r')(s + s') = rs + r's + rs' + r's' \text{ for all } r, r', s, s' \in R.$$

- A ring is called commutative if:

$$rs = sr \text{ for all } r, s \in R.$$

- A division ring is a ring D, such that $D \setminus \{0\}$ is a multiplicative group.
- A field is a commutative division ring.
- A subring of a ring R, is an additive subgroup S, that is also closed with respect to the multiplication rule on R and contains $1 \in R$.

- The centre of a ring R, denoted $\mathcal{Z}(R)$, is the set of all elements $r \in R$ such that $rs = sr$ for all $s \in R$. This can easily be seen to be a subring of R.

- An algebra is a ring R that contains a field as a central subring. If F is that field, we often refer to R as an F–algebra.

- A left module M, over a ring R, is an additive abelian group equipped with a scalar product $(-).(-) : R \times M \longrightarrow M$ satisfying:

$$1.m = m \text{ for all } m \in M;$$
$$(rs).m = r.(s.m) \text{ for all } r, s \in R, m \in M;$$
$$(r + r').(m + m') = r.m + r.m' + r'.m + r'.m' \text{ for all } r, r' \in R, m, m' \in M.$$

  A right R–module is defined analogously. We will not normally write the dots in the scalar product.

- An R–submodule of an R–module M is an additive subgroup that is closed under the scalar product on M. A nonzero module with no nontrivial submodules (i.e. the only submodules are itself and $\{0\}$) is called simple.

- A left ideal of a ring R is an additive subgroup that is also a left module with scalar product given by the multiplication law on R. A right ideal is defined analogously. An additive subgroup that is both a left and right ideal is called a two–sided ideal. We will normally assume that "ideal" means a two–sided ideal.

1.1.2 Definition

A two–sided proper (i.e. not the whole ring) ideal P, is called prime if for any two ideals I and J, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$. If the zero ideal is a prime ideal the ring is called a prime ring. The set of all prime ideals of a ring R is denoted $\text{spec}(R)$, its prime spectrum.

1.1.3 Lemma

The following are equivalent:

- *P is a prime ideal.*
- *$\frac{R}{P}$ is a prime ring.*
- *For any two left ideals I and J, $IJ \subseteq P \Rightarrow I \subseteq P or J \subseteq P$.*
- *The right handed analogue of the above condition.*
- *For any two elements x and y in R, $xRy \subseteq P \Rightarrow x \in P$ or $y \in P$.*

Proof:

See [Goo92, Prop 2.1].

1.1.4 Lemma

Let X be a subset of a ring, R, such that $0 \notin X$ and X is closed under multiplication. Let P be a two–sided ideal maximal amongst ideals disjoint from X, then P is prime.

Proof:

See [Goo92, Lemma 2.5].

1.1.5 Definition

An ideal Q is primitive if $Q = \text{ann}(M) := \{q \in R : qM = 0\}$ for some simple R–module M. If the zero ideal is primitive, R is called a primitive ring. The set of all primitive ideals is denoted $\text{prim}(R)$.

1.1.6 Remark

Strictly speaking, we have defined *left primitivity.* There is an analogous definition for *right primitivity.* We shall not make much distinction between the two as it will always be clear how to adapt left handed results to give right handed ones. Nevertheless, it is important to note that a left primitive ideal may not necessarily be right primitive as shown by an example of Bergman.

1.1.7 Lemma

Every primitive ideal is prime. Every maximal ideal is primitive.

Proof:

See [Goo92, Prop 2.15]. //

1.2 Special elements of a ring

Here we recall the definitions of certain properties an element of a ring might have.

1.2.1 Definition

- An element $r \in R$ is normal if $rR = Rr$.
- An element $r \in R$ is a unit if there exists $r^{-1} \in R$ such that $rr^{-1} = r^{-1}r = 1$.
- An element $r \in R$ is a zerodivisor if there exists $s \in R$ such that either $rs = 0$ or $sr = 0$, otherwise r is a nonzerodivisor.

1.3 Noetherian rings

Here we recall the definition of noetherianity for a ring.

1.3.1 Definition

A ring R is left noetherian if there are no infinite strictly ascending chains:

$$I_0 \subset I_1 \subset I_2 \subset I_3 \subset \dots$$

of left ideals, I_i. *Right noetherianity* is defined analogously. A ring which is both left and right noetherian is called a noetherian ring.

1.4 Filtrations and gradings

Here we describe various ways of equipping a ring with a notion of degree.

1.4.1 Definitions

- A ring R is called filtered if $R = \bigcup_{i\in\mathbb{Z}^+} R_i$ where R_i are additive subgroups of R with the properties:
  - $R_i \subseteq R_{i+1}$;
  - $R_iR_j \subseteq R_{i+j}$.
- A ring R is called graded if $R = \bigoplus_{i\in\mathbb{Z}^+} R_i$ where R_i are additive subgroups of R with the property:
$$R_iR_j \subseteq R_{i+j}.$$
- More generally, given an abelian group, G, a ring is G–graded if $R = \bigoplus_{g\in G} R_g$ where R_g are additive subgroups of R with the property:
$$R_gR_h \subseteq R_{gh}.$$

1.4.2 Lemma

Let R be a filtered ring then $\operatorname{gr} R := \bigoplus_{i\in\mathbb{Z}^+} \frac{R_i}{R_{i-1}}$ (with the convention $R_{-1} = 0$) is a graded ring with the multiplication given by $(r + R_{i-1})(s + R_{j-1}) = rs + R_{ij-1}$.

Proof:

See [MR87, §1.6.4]. //

1.4.3 Lemma

Let R be a filtered ring.

- *If $\operatorname{gr} R$ is a domain, then so is R.*

- *If* $\operatorname{gr} R$ *is prime, then so is* R.
- *If* $\operatorname{gr} R$ *is noetherian, then so is* R.

Proof:

See [MR87, Props 1.6.6&1.6.9]. //

1.4.4 Definitions

- An ideal I, of a graded ring R, is graded if whenever $r \in I$ and $r = \sum_{i=0}^{n} r_i$ where $r_i \in R_i$ we have $r_i \in I$ for all i.
- An ideal, P, is gr–prime if, for any two graded ideals I and J, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

1.5 Localisation

It is often convenient to adjoin inverses of certain elements to a ring we are studying.

1.5.1 Definition

Let $\mathbb{X}$ be a subset of a ring R satisfying the following:

- $\mathbb{X}$ is closed under multiplication.
- $1 \in \mathbb{X}$.
- For any $x \in \mathbb{X}, r \in R$ there exists $x' \in \mathbb{X}, r' \in R$ such that $rx' = xr'$.

Then $\mathbb{X}$ is called a right Ore set. There is an analogous concept of a *left Ore set*. If we have that x is a nonzerodivisor for all $x \in \mathbb{X}$, then $\mathbb{X}$ is called a right denominator set this is slightly stronger than the standard definition (given in [MR87], say) and is assumed for convenience.

1.5.2 Lemma

Let $\mathbb{X}$ be a right denominator set in R. Then there exists a right localisation *$R\mathbb{X}^{-1}$ which is universal among codomains of homomorphisms on R with the property that elements of $\mathbb{X}$ map to units. Moreover, $R \hookrightarrow R\mathbb{X}^{-1}$ under the localisation homomorphism.*

Proof:

See [MR87, §2]. //

We shall identify R with its image in $R\mathbb{X}^{-1}$.

1.5.3 Lemma

Let $\mathbb{X}$ be a right denominator set in a noetherian ring R. Then the maps $P \longrightarrow P\mathbb{X}^{-1}$ and $Q \longrightarrow Q \bigcap R$ are mutually inverse bijections between $\{P \in \mathrm{spec}(R) : P \bigcap \mathbb{X} = \emptyset\}$ and $\{Q \in \mathrm{spec}(R\mathbb{X}^{-1})\}$.

Proof:

See [MR87, Prop 2.1.16]. //

1.5.4 Lemma

Let $\mathbb{X}$ be a right denominator set in a ring, R, let M be a (right) R–module, let $T(M) := \{m \in M : mx = 0 \text{ for some } x \in \mathbb{X}\}$ and let $\overline{M} := \frac{M}{T(M)}$ then $\overline{M}\mathbb{X}^{-1}$ is a right $R\mathbb{X}^{-1}$ module with the multiplication given by using the Ore condition to move the denominators to the extreme right then carrying out the module multiplication as in M.

Proof:

See [MR87, §1.17]. //

1.6 Skew Polynomial Rings

Diffusion algebras turn out to arise from iterated applications of the following standard construction.

1.6.1 Definition

Let R be a ring, α an endomorphism of R and δ a right α–derivation (i.e. an additive group endomorphism of R with the property $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for all $r, s \in R$). Then $R[x; \alpha, \delta]$ denotes the right skew polynomial ring. As a left R–module it is free on the set $\{1, x, x^2, \dots\}$ and its multiplication law is inherited from the rule

$$xr = \alpha(r)x + \delta(r) \text{ for all } r \in R.$$

There is an analogous concept of a *left skew polynomial ring*, written $[\delta, \alpha; x]R$, where δ is a *left α–derivation*.

1.7 Stratification

We now let an algebraic group (see [Spr81, §2.1.1] for the definition) H act on a ring R by automorphisms. A *stratification* is a decomposition of the prime and primitive spectra motivated by this action. See [BG02] for more details.

1.7.1 Definitions

- An ideal I is an H–ideal if $H(I) \subseteq I$.
- An H–ideal P is H–prime if, for any H–ideals I and J, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$. The set of all H–primes is denoted $H - \operatorname{spec} R$.
- Let P be a prime ideal, the ideal $(P : H) := \bigcap_{Q=h(P), h\in H} Q$ is H–prime (see [BG02, II.1.9].
- Let J be an H–prime. Then $\operatorname{spec}_J R := \{P \in \operatorname{spec} R : (P : H) = J\}$, $\operatorname{prim}_J R := \{Q \in \operatorname{prim} R : (Q : H) = J\}$.

1.7.2 Lemma

Let A be a noetherian k–algebra where k is an infinite field, let $H := (k^\star)^r$ act rationally (see [BG02, II.2.6]) on A by k–algebra automorphisms and let $J \in H - \operatorname{spec} R$. Then

- *J is prime*
- *The set $\mathbb{X}_J$ of all H–eigenvectors in $\frac{A}{J}$ that are nonzerodivisors is a denominator set. Moreover $A_J := \frac{A}{J}\mathbb{X}_J^{-1}$ is H–simple (i.e. 0 is the only H–prime).*
- *$\operatorname{spec}_J A \cong \operatorname{spec} A_J$ via localisation and contraction.*
- *$\operatorname{spec} A_J \cong \operatorname{spec} \boldsymbol{\mathcal{Z}}(A_J)$ via extension contraction.*
- *$\boldsymbol{\mathcal{Z}}(A_J)$ is a Laurent polynomial ring in at most r indeterminates over the fixed field $\boldsymbol{\mathcal{Z}}(A_J)^H$.*

Proof:

See [BG02, II.2.13]. //

Note that, for a $\mathbb{Z}^r$ grading, after [BG02, II.2.11], gr–prime ideals are prime.

1.7.3 Lemma

Let A be a noetherian k–algebra, where k is an algebraically closed field. Let H be an affine algebraic group over k acting rationally on A by k–algebra automorphisms. If $H - \operatorname{spec} A$ is finite and J is H–prime then $\operatorname{prim}_J R = \{Q \in \operatorname{spec}_J R : Q \text{ is maximal in } \operatorname{spec}_J R\}$.

Proof:

See [BG02, II.8.10]. //

1.8 The Diamond Lemma

Consider a presentation of an algebra $\frac{k\langle X\rangle}{I}$ where X is a set of generators, $k\langle X\rangle$, the free k–algebra on X and I is an ideal.

Well-order the monomials in $k\langle X\rangle$ by total degree (choosing a lexicographic ordering on X to break ties).

Choose a set of generators g_j for I. We rewrite the relations $g_j = 0$ as $w_j = f_j$ where w_j is the biggest monomial appearing in g_j with respect to our ordering and $f_j = w_j - g_j$. Assume, without loss of generality, that the coefficient of w_j is 1.

1.8.1 Definitions

- The k–linear map $r_{ajb} : k\langle X\rangle \rightarrowtail k\langle X\rangle$ defined for any monomials a and b by sending aw_jb to af_jb and fixing all other monomials is called an <u>elementary reduction</u>.
- A 5–tuple $(a, b, c, w_{j_1}, w_{j_2})$ of monomials is called an <u>overlap ambiguity</u> if $abc = w_{j_1}c = aw_{j_2}$. It is <u>resolvable</u> if there exist compositions r_1 and r_2 of elementary reductions such that $r_1(f_{j_1}c) = r_2(af_{j_2})$.
- A 5–tuple $(a, b, c, w_{j_1}, w_{j_2})$ of monomials is called an <u>inclusion ambiguity</u> if $abc = aw_{j_1}c = w_{j_2}$. It is <u>resolvable</u> if there exists compositions, r_1 and r_2, of elementary reductions such that $r_1(af_{j_1}c) = r_2(f_{j_2})$.

1.8.2 Lemma

If all ambiguities are resolvable, then the set of monomials in $k\langle X\rangle$ that are irreducible (i.e. any elementary reduction acts trivially on them) maps to a basis of the algebra $\frac{k\langle X\rangle}{I}$.

Proof:

See [Ber78, Thm 1.2]. //

Chapter 2

Diffusion Algebras

Diffusion algebras arise in physics as a potential means to understand a seemingly large class of 1–dimensional stochastic processes (see [IPR01] and references therein). We recall the definition of a diffusion algebra and introduce our various notations. We will also record some lemmas on bases and dimension which, although trivial, are helpful.

2.1 Definitions and Bases

2.1.1 Definition

Let R be the algebra generated by n indeterminates, x_1, x_2, $\dots$, x_n over $\mathbb{C}$ subject to relations:

$$a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j, \tag{2.1.1.1}$$

whenever $i < j$ for some parameters $a_{ij} \in \mathbb{C} \setminus \{0\}$ for all $i < j$ and $b_{ij}, r_i \in \mathbb{C}$ for all $i < j$. We define the standard monomials to be those of the form $x_n^{i_n} x_{n-1}^{i_{n-1}} \dots x_2^{i_2} x_1^{i_1}$. R is called a diffusion algebra if it admits a *PBW–basis of these standard monomials.* In other words, R is a diffusion algebra if these standard monomials are a $\mathbb{C}$–vector space basis for R.

In the applications to physics, the parameters a_{ij} are strictly positive reals and the parameters b_{ij} are positive reals as they are unnormalised measures of probability. We will denote $q_{ij} := \frac{b_{ij}}{a_{ij}}$. The only way a parameter q_{ij} in a diffusion algebra arising from physics can be a root of unity is if it is 1. It is therefore reasonable that we will sometimes assume these parameters not to be a root of unity other than 1.

An n–generator diffusion algebra R, has Gelfand–Kirillov dimension n (as defined in [MR87, ch. 8]) since, because of the PBW–basis, the vector subspace consisting of elements of total degree at most k is isomorphic to that of a commutative polynomial ring in n indeterminates. Thus n is an invariant of the algebra.

2.1.2 Remarks

- A diffusion algebra in one indeterminate is just a commutative polynomial ring in one indeterminate. An algebra in two indeterminates subject to (2.1.1.1) is always a diffusion algebra.
- Recall the definition of multiparameter quantum affine n–space. This is the $\mathbb{C}$–algebra generated by n indeterminates, $x_1, x_2, \ldots, x_n$, subject to the relations:

$$x_i x_j - q_{ij} x_j x_i = 0 \quad \text{whenever } i < j,$$

  where $q_{ij} \in \mathbb{C} \backslash \{0\}$. This algebra will occur often in our analysis of diffusion algebras. For example, a diffusion algebra with $r_i = 0$ for all i is a multiparameter quantum affine n–space.

2.1.3 Lemma

An algebra generated by x_1,x_2 and x_3 satisfying the three relations (2.1.1.1) *is a diffusion algebra if and only if the parameters satisfy:*

$$b_{12}b_{13}r_2 - a_{23}b_{13}r_2 = -b_{23}b_{13}r_2 + a_{12}b_{13}r_2; \tag{2.1.3.1a}$$

$$b_{12}b_{13}r_3 + a_{23}b_{12}r_3 = b_{12}b_{23}r_3 + a_{13}b_{12}r_3; \tag{2.1.3.1b}$$

$$r_1b_{12}b_{23} + a_{13}r_1b_{23} = r_1b_{13}b_{23} + a_{12}r_1b_{23}; \tag{2.1.3.1c}$$

$$r_2r_3b_{23} + a_{13}r_2r_3 - a_{12}r_2r_3 = a_{23}r_2r_3; \tag{2.1.3.1d}$$

$$r_1r_2b_{12} - a_{23}r_1r_2 + a_{13}r_1r_2 = a_{12}r_1r_2; \tag{2.1.3.1e}$$

$$r_1r_3b_{12} = r_1r_3b_{23}. \tag{2.1.3.1f}$$

(2.1.3.1)

Conditions on the parameters to guarantee the PBW–basis first appeared in [IPR01, eqns (2.5)–(2.10)] with different notation.

Proof:

It suffices, by the diamond lemma (1.8.2), to prove that (2.1.3.1) are equivalent to the unique reduction of the monomial $x_1x_2x_3$ to a linear combination of standard monomials. We do this by matching coefficients in the two reductions $x_1(x_2x_3)$ and $(x_1x_2)x_3$. We show only the coefficient of x_2x_1 as an example, the others are no different. The coefficient of x_2x_1 in $x_1(x_2x_3)$ is $\frac{b_{12}b_{23}r_3}{a_{12}a_{13}a_{23}} + \frac{b_{12}r_3}{a_{12}a_{23}}$, the same coefficient in $(x_1x_2)x_3$ is $\frac{b_{12}b_{13}r_3}{a_{12}a_{13}a_{23}} + \frac{b_{12}r_3}{a_{12}a_{13}}$, and these are equal if and only if (2.1.3.1b) holds. //

2.1.4 Remarks

(i) Using our q_{ij} notation, (2.1.3.1) can be rewritten as:

$$
\begin{aligned}
q_{13}r_2\left(a_{12}(q_{12}-1) + a_{23}(q_{23}-1)\right) &= 0; && \text{(2.1.4.1a)}\\
q_{12}r_3\left(a_{13}(q_{13}-1) - a_{23}(q_{23}-1)\right) &= 0; && \text{(2.1.4.1b)}\\
q_{23}r_1\left(a_{12}(q_{12}-1) - a_{13}(q_{13}-1)\right) &= 0; && \text{(2.1.4.1c)}\\
r_2r_3\left(a_{23}(q_{23}-1) + a_{13} - a_{12}\right) &= 0; && \text{(2.1.4.1d)}\\
r_1r_2\left(a_{12}(q_{12}-1) - a_{23} + a_{13}\right) &= 0; && \text{(2.1.4.1e)}\\
r_1r_3\left(a_{12}q_{12} - a_{23}q_{23}\right) &= 0. && \text{(2.1.4.1f)}
\end{aligned}
\tag{2.1.4.1}
$$

(ii) An n–generator algebra subject to (2.1.1.1) is a diffusion algebra if and only if the conditions analogous to (2.1.4.1) are satisfied for any 3–generator subalgebra. This again follows from the diamond lemma.

2.1.5 Lemma

Let R be a 3–generator diffusion algebra, then:

(i) *If $q_{12} \neq 0$, R also has a PBW–basis of monomials $x_3^{i_3}x_1^{i_2}x_2^{i_1}$.*

(ii) *If* $q_{23} \neq 0$, *R also has a PBW–basis of monomials* $x_2^{i_3} x_3^{i_2} x_1^{i_1}$.

(iii) *An n–generator diffusion algebra with PBW–basis* $x_n^{i_n} \dots x_{j+1}^{i_{j+1}} x_j^{i_j} \dots x_2^{i_2} x_1^{i_1}$ *is also a diffusion algebra with PBW–basis* $x_n^{i_n} \dots x_j^{i_j} x_{j+1}^{i_{j+1}} \dots x_2^{i_2} x_1^{i_1}$ *if* $q_{j(j+1)} \neq 0$, *where* $1 \leqslant j < n$.

Proof:

We will prove only the first part (the second part is similar and the third part follows using the diamond lemma (1.8.2) and one of the first two parts on an appropriate 3–generator subalgebra). Observe that, if $q_{12} \neq 0$, then changing the order of generators $x'_1 := x_2$, $x'_2 := x_1$ and $x'_3 := x_3$ preserves (2.1.1.1) with $a'_{12} = b_{12}$, $a'_{13} = a_{23}$, $a'_{23} = a_{13}$, $b'_{12} = a_{12}$, $b'_{13} = b_{23}$, $b'_{23} = b_{13}$, $r'_1 = r_2$, $r'_2 = r_1$ and $r'_3 := r_3$. The conditions (analogous to (2.1.4.1)) to have a PBW–basis with the new order follow easily from the conditions, (2.1.4.1), to have a PBW–basis in the original order. //

2.1.6 Remark

It follows that, if all the q_{ij}s are nonzero, we have a PBW–basis in any order.

2.1.7 Lemma

The opposite ring of a 3*–generator (or indeed, n–generator) diffusion algebra (PBW–basis* $x_3^{i_3} x_2^{i_2} x_1^{i_1}$*) is also a diffusion algebra (with PBW–basis* ${x_1^{\mathrm{op}}}^{i_1} {x_2^{\mathrm{op}}}^{i_2} {x_3^{\mathrm{op}}}^{i_3}$*).*

Proof:

The opposite ring of a diffusion algebra satisfies:

$$(a_{ij} x_i x_j - b_{ij} x_j x_i)^{\mathrm{op}} = (r_j x_i - r_i x_j)^{\mathrm{op}},$$

whenever $i < j$, i.e.

$$a_{ij} x_j^{\mathrm{op}} x_i^{\mathrm{op}} - b_{ij} x_i^{\mathrm{op}} x_j^{\mathrm{op}} = r_j x_i^{\mathrm{op}} - r_i x_j^{\mathrm{op}},$$

thus, if we consider our monomials to be ordered the other way around (i.e. rename $x'_n := x_1^{\mathrm{op}}, \dots, x'_2 := x_{n-1}^{\mathrm{op}}, x'_1 := x_n^{\mathrm{op}}$), these relations are (2.1.1.1) (with $r'_i := -r_i$ for all i).

As a $\mathbb{C}$–vector space, $R^{\text{op}} \cong R$ thus if R has a $\mathbb{C}$–vector space basis of monomials $x_3^{i_3}x_2^{i_2}x_1^{i_1}$, then R^{op} has a $\mathbb{C}$–vector space basis of elements $\left(x_3^{i_3}x_2^{i_2}x_1^{i_1}\right)^{\text{op}}$ which are precisely the monomials $x_1^{\text{op}^{i_1}}x_2^{\text{op}^{i_2}}x_3^{\text{op}^{i_3}}$. //

2.2 Noetherianity

We shall now identify precisely when an n–generator diffusion algebra is noetherian in terms of the parameters q_{ij}. The trivial case of a 1–generator diffusion algebra is always noetherian as it is always a polynomial ring in one variable (it also has no parameters q_{ij}). Noetherianity of diffusion algebras in two or more generators will be approached by viewing them as iterated skew polynomial rings.

2.2.1 Lemma

A 3–generator diffusion algebra (generated by x_1, x_2 and x_3 as above) is a right skew polynomial ring over its 2–generator diffusion subalgebra generated by x_2 and x_3.

Proof:

Define an endomorphism, α, and α–derivation, δ, on the free algebra generated by x_2 and x_3 by:

$$\begin{aligned} \alpha(x_2) &= q_{12}x_2 + \frac{r_2}{a_{12}}; \quad \delta(x_2) = -\frac{r_1}{a_{12}}x_2; \\ \alpha(x_3) &= q_{13}x_3 + \frac{r_3}{a_{13}}; \quad \delta(x_3) = -\frac{r_1}{a_{13}}x_3. \end{aligned} \tag{2.2.1.1}$$

It follows from (2.1.3.1) that the defining relations (2.1.1.1), for $i = 2$ and $j = 3$, are in the kernels of these functions and thus they induce an endomorphism (also denoted α) and α–derivation (also denoted δ) of the diffusion subalgebra generated by x_2 and x_3. Thus we can construct the skew polynomial ring adjoining x_1 to the diffusion algebra generated by x_2 and x_3, twisting by α and δ. This skew polynomial ring satisfies (2.1.1.1), thus there is a canonical surjective homomorphism from the diffusion algebra generated by x_1,

x_2 and x_3 onto it. This is an isomorphism because the skew polynomial ring also has a PBW–basis of monomials $x_3^{i_3}x_2^{i_2}x_1^{i_1}$. //

2.2.2 Remarks

- It is easy to show that a 2–generator diffusion algebra is a right skew polynomial ring over the polynomial subalgebra generated by x_2.
- Moreover, an n–generator diffusion algebra (generated by $x_1, \ldots, x_n$ as above) is a right skew polynomial ring over its $(n-1)$–generator diffusion subalgebra generated by $x_2, \ldots, x_n$ (in fact an n–generator diffusion algebra is an iterated skew polynomial ring in the obvious way).

2.2.3 Lemma

For any ring S, any endomorphism, α, any α–derivation, δ, and any $a \in S$, we have $S[x;\alpha,\delta] \cong S[(x-a);\alpha,\delta']$ where $\delta'(s) := \delta(s) + \delta_a(s)$, where δ_a is the inner α–derivation given by a.

Proof:

This result is a minor generalisation of the well known result that, for δ an inner α–derivation, $R[X;\alpha,\delta] \cong R[X';\alpha]$ where X and X' are indeterminates, see [Goo92, Lem. 1.5c]. The proof is similar. //

2.2.4 Lemma

For any ring S, any endomorphism, α, of S and any α–derivation, δ, of S, $S' := S[x;\alpha,\delta]$ is not left noetherian if x is a zerodivisor.

Proof:

There must exist $h \in S' \setminus \{0\}$ such that $xh = 0$ ($hx \neq 0$ since S' is a free left S–module on $1,\ x,\ x^2,\ \ldots$). Then $H_n := Sh + Shx + \cdots + Shx^n$ is an infinite, strictly ascending chain of left ideals of S'. It is clear that H_n is closed under addition and left multiplication by elements of S. It suffices to show that $xH_n \subseteq H_n$. This follows from the fact that $xS \subseteq Sx + S$ and that $xh = 0$, thus:

$$\begin{aligned} xH_n &= xSh + xShx + \cdots + xShx^n \\ &\subseteq (Sx+S)h + (Sx+S)hx + \cdots + (Sx+S)hx^n \\ &= Sh + Shx + \cdots + Shx^n = H_n. \end{aligned}$$

The chain of left ideals H_n is strictly ascending since, if we filter S' by letting $\deg x := 1$ and $\deg S := 0$, then the maximum degree of an element of H_n is $n + \deg h$ as S' is a free right S–module with basis $\{1, x, x^2, \dots\}$. //

2.2.5 Proposition

A diffusion algebra on $n \geqslant 2$ generators is left noetherian if and only if we have $q_{ij} \neq 0$ for all $i < j$.

Proof:

Observe first that if we equip our n-generator diffusion algebra, R, with the standard filtration ($\deg(x_i) = 1$ for all i) and assume that all the q_{ij}s are nonzero then $\operatorname{gr} R$ is isomorphic to multiparameter quantum affine n–space which is known to be noetherian, thus so is R (by (1.4.3)). Alternatively, present R as a skew polynomial ring as in (2.2.1) and use Hilbert's basis theorem [GW89, §1.2.9(iv)] (where all the endomorphisms are bijective since all the q_{ij}s are nonzero).

We now assume that at least one $q_{ij} = 0$ and show that R is not left noetherian. Without loss of generality, we may assume that $q_{1j} = 0$ as we may choose the least i for which $q_{ij} = 0$ for some j; we may then exchange generators x_1 and x_i to get an isomorphic diffusion algebra with $q_{1j} = 0$ (see (2.1.5)). This can be viewed as a skew polynomial ring (attaching the new indeterminate x_1) over the diffusion subalgebra generated by $x_2, \dots, x_n$. We may consider this to be a skew polynomial ring attaching the indeterminate $x_1' := x_1 + \frac{r_1}{a_{1j}}$, by (2.2.3). By (2.1.1.1), we have:

$$x_1' x_j = \frac{r_j}{a_{1j}} x_1' - \frac{r_1 r_j}{a_{1j}}.$$

If $r_1 r_j = 0$, then $x_1'(x_j - \frac{r_j}{a_{1j}}) = 0$; otherwise $x_1'((x_j - \frac{r_j}{a_{1j}})x_1' + \frac{r_1 r_j}{a_{1j}}) = 0$. In both cases x_1' is a zero divisor therefore, by (2.2.4), R is not left noetherian. //

2.2.6 Remarks

- The right–handed analogue of (2.2.4) is also true. Note that, in this analogue, the indeterminate is attached on the left instead of on the right as usual. Viewing our diffusion algebra as an iterated skew polynomial ring with each new indeterminate attached on the left, we may also conclude: A diffusion algebra is right noetherian if and only if $q_{ij} \neq 0$ for all $i < j$.
- A diffusion algebra is left (right) noetherian if and only if it is a domain.
- A noetherian diffusion algebra is Auslander–regular and Cohen–Macaulay, by iterated applications of [LS93, lemma].
- By (2.1.6) a noetherian diffusion algebra has a PBW–basis of standard monomials whatever order we give to the indeterminates (i.e. the monomials of the form $x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} \dots x_{\sigma(n)}^{i_n}$ form a $\mathbb{C}$–basis of an n–generator noetherian diffusion algebra for any fixed choice of $\sigma \in S_n$).

2.2.7 Corollary

If R is a non–noetherian 3–generator diffusion algebra then, without loss of generality, $q_{12} = 0$ (in the sense that either R or R^{op} can be viewed as a diffusion algebra of this kind).

Proof:

If R is non–noetherian then at least one of the q_{ij}s is zero. If $q_{12} = 0$ then we are done. If $q_{23} = 0$ then we may view R^{op} as a diffusion algebra (using (2.1.7)) with $q_{12} = 0$. If $q_{12} \neq 0 \neq q_{23}$ then we must have $q_{13} = 0$; exchanging generators $x_3' := x_2, x_2' := x_3$ we may view R as a diffusion algebra with $q_{12} = 0$. //

Chapter 3

Noetherian Diffusion Algebras

3.1 Classification

We will now show that every noetherian diffusion algebra is isomorphic to one of three different types of algebra. Throughout this chapter, R will be a diffusion algebra generated by $n \geqslant 2$ indeterminates $(x_1, x_2, \ldots, x_n)$ with, of course, $q_{ij} \neq 0$ for all i, j.

We shall, as motivation, first give a complete classification of 2–generator noetherian diffusion algebras which will, later, be subsumed into our two main propositions that constitute the n–generator classification:

3.1.1 Proposition

Every 2–generator noetherian diffusion algebra is isomorphic to one of the following three types of algebra:

- The quantum affine plane,
  (generated by X_1 and X_2, subject to the relation $X_1X_2 - qX_2X_1 = 0$ for some $q \in \mathbb{C} \setminus \{0\}$, allowing the possibility $q = 1$)

- The *quantised Weyl algebra*,
  (generated by X_1 and X_2, subject to the relation $X_1X_2 - qX_2X_1 = 1$ for some $q \in \mathbb{C} \setminus \{0, 1\}$)

- The *universal enveloping algebra of the* 2–d *soluble Lie algebra*, (generated by X_1 and X_2, subject to the relation $X_1X_2 - X_2X_1 = X_1$)

Proof:

Since $a_{12} \neq 0 \neq b_{12}$, without loss of generality, our diffusion algebra is generated by x_1 and x_2 subject to the relation:

$$x_1x_2 - qx_2x_1 = r_2x_1 - r_1x_2.$$

If $q \neq 1$, we let $X_1 := x_1 + \frac{r_1}{1-q}$ and $X_2 := x_2 - \frac{r_2}{1-q}$ and we have:

$$X_1X_2 - qX_2X_1 = \frac{r_1r_2}{1-q},$$

which is either an affine quantum plane (if $r_1 = 0$ or $r_2 = 0$) or a quantised Weyl algebra (if $r_1 \neq 0 \neq r_2$, after rescaling to get a 1 on the right hand side). If $q = 1$ then either we have a commutative polynomial ring in two variables (which we consider to be an affine quantum plane, occurring if $r_1 = r_2 = 0$), or we can assume either r_1 or r_2 is nonzero. We show only the case $r_1 \neq 0$: letting $X_1 := r_2x_1 - r_1x_2$ and $X_2 := \frac{x_2}{r_2}$, we have:

$$X_1X_2 - X_2X_1 = X_1.$$

//

3.1.2 Remarks

- Observe that it seems a quite important distinction whether a parameter q_{ij} is one or not: If $q_{ij} = 1$ then the only possible candidates for the 2–generator subalgebra generated by x_i and x_j are a quantum affine plane and a 2–d universal enveloping algebra; if $q_{ij} \neq 1$ only quantum affine planes and quantised Weyl algebras are possible. Moreover, the change of variables used to reduce to one of the three algebras differs between these two cases.

- Since we are assuming all the q_{ij}s to be nonzero, we may freely permute our generators (see (2.1.5)) and our conditions on the parameters for a PBW–basis become:

$$r_\beta\left(a_{\alpha\beta}(q_{\alpha\beta}-1)+a_{\beta\gamma}(q_{\beta\gamma}-1)\right)=0; \tag{3.1.2.1a}$$
$$r_\gamma\left(a_{\alpha\gamma}(q_{\alpha\gamma}-1)-a_{\beta\gamma}(q_{\beta\gamma}-1)\right)=0; \tag{3.1.2.1b}$$
$$r_\alpha\left(a_{\alpha\beta}(q_{\alpha\beta}-1)-a_{\alpha\gamma}(q_{\alpha\gamma}-1)\right)=0; \tag{3.1.2.1c}$$
$$r_\beta r_\gamma\left(a_{\beta\gamma}(q_{\beta\gamma}-1)+a_{\alpha\gamma}-a_{\alpha\beta}\right)=0; \tag{3.1.2.1d}$$
$$r_\alpha r_\beta\left(a_{\alpha\beta}(q_{\alpha\beta}-1)-a_{\beta\gamma}+a_{\alpha\gamma}\right)=0; \tag{3.1.2.1e}$$
$$r_\alpha r_\gamma\left(a_{\alpha\beta}(q_{\alpha\beta})-a_{\beta\gamma}(q_{\beta\gamma})\right)=0, \tag{3.1.2.1f}$$

(3.1.2.1)

for any triple $\alpha<\beta<\gamma\in\{1,2,\ldots,n\}$.

3.1.3 Lemma

If $q_{ij}\notin\{0,1\}$ for all i,j then at most two of the parameters r_i are nonzero.

Proof:

Assume that r_α, r_β and r_γ are nonzero (without loss of generality $\alpha<\beta<\gamma$). Combining (3.1.2.1a) and (3.1.2.1b) gives us that $a_{\alpha\beta}(q_{\alpha\beta}-1)=-a_{\alpha\gamma}(q_{\alpha\gamma}-1)$. However, by (3.1.2.1c), $a_{\alpha\beta}(q_{\alpha\beta}-1)=a_{\alpha\gamma}(q_{\alpha\gamma}-1)$. This is a contradiction since $a_{\alpha\gamma}\neq 0$ and $q_{\alpha\gamma}\neq 1$. //

3.1.4 Proposition

If $q_{ij}\notin\{0,1\}$ for all i,j then R is isomorphic either to multiparameter quantum affine n–space or to the $\mathbb{C}$–algebra generated by X_1, X_2, $x_3,\ldots,x_n$ subject to relations:

$$X_1X_2-q_{12}X_2X_1=1 \qquad \text{where } q_{12}\neq 1; \tag{3.1.4.1a}$$
$$X_1x_i-q_{1i}x_iX_1=0 \qquad \text{where } q_{1i}\neq 1; \tag{3.1.4.1b}$$
$$X_2x_i-q_{1i}^{-1}x_iX_2=0; \tag{3.1.4.1c}$$
$$x_ix_j-q_{ij}x_jx_i=0 \qquad \text{for all } 3\leqslant i<j. \tag{3.1.4.1d}$$

(3.1.4.1)

N.B. Algebras of type (3.1.4.1) with some parameters $q_{ij} = 1$ for $i \geqslant 3$ will arise later.

Proof:

Without loss of generality, our diffusion algebra, R, (generated by $x_1, x_2, \ldots, x_n$) has $r_i = 0$ for all $i \geqslant 3$. Our conditions (3.1.2.1) on the parameters for R to have a PBW–basis thus become:

$$a_{12}(q_{12} - 1) + a_{2i}(q_{2i} - 1) = 0 \quad \text{if } r_2 \neq 0; \tag{3.1.4.2a}$$

$$a_{12}(q_{12} - 1) - a_{1i}(q_{1i} - 1) = 0 \quad \text{if } r_1 \neq 0; \tag{3.1.4.2b}$$

$$a_{12}(q_{12} - 1) - a_{2i} + a_{1i} = 0 \quad \text{if } r_2 \neq 0 \text{ and } r_2 \neq 0, \tag{3.1.4.2c}$$

for all $i \geqslant 3$ and:

$$a_{\alpha i}(q_{\alpha i} - 1) - a_{\alpha j}(q_{\alpha j} - 1) = 0 \quad \text{if } r_\alpha \neq 0, \tag{3.1.4.2d}$$

for all $\alpha \in \{1, 2\}$ and for all $3 \leqslant i < j$,

(3.1.4.2)

where the first three come from considering (3.1.2.1) for any subalgebra generated by x_1, x_2 and x_i (where $i \in \{3, \ldots, n\}$) and the last comes from considering a subalgebra generated by x_α, x_i and x_j (where $\alpha \in \{1, 2\}$, $i < j \in \{3, \ldots, n\}$). Note that (3.1.4.2d) follows from (3.1.4.2a) and (3.1.4.2b).

We make the change of generators:

$$x_1' := x_1 + \frac{r_1}{a_{12}(1 - q_{12})} \overset{(3.1.4.2b)}{=} x_1 + \frac{r_1}{a_{1i}(1 - q_{1i})} \text{ for all } i \geqslant 3;$$
$$x_2' := x_2 - \frac{r_2}{a_{12}(1 - q_{12})} \overset{(3.1.4.2a)}{=} x_2 + \frac{r_2}{a_{2i}(1 - q_{2i})} \text{ for all } i \geqslant 3.$$

Therefore our algebra is now generated by x'_1, x'_2, $x_3, \ldots, x_n$ subject to relations:

$$\begin{aligned} a_{12}\left(x'_1x'_2 - q_{12}x'_2x'_1\right) &= -\frac{r_1r_2}{a_{12}(1-q_{12})}; \\ x_\alpha x_i - q_{\alpha i}x_ix_\alpha &= 0 \quad \text{for all } \alpha \in \{1,2\}, i \in \{3,\ldots,n\}; \\ x_ix_j - q_{ij}x_jx_i &= 0 \quad \text{for all } i < j \in \{3,\ldots,n\}. \end{aligned}$$

If either $r_1 = 0$ or $r_2 = 0$ then this is a quantum n–space. Otherwise our algebra satisfies (3.1.4.1b) and (3.1.4.1d). Observe that substituting (3.1.4.2b) into (3.1.4.2c) gives:

$$\frac{q_{1i}}{a_{12}a_{2i}} = \frac{1}{a_{12}a_{1i}}$$

and substituting (3.1.4.2a) into (3.1.4.2c) gives:

$$\frac{q_{2i}}{a_{12}a_{1i}} = \frac{1}{a_{12}a_{2i}},$$

thus $q_{1i} = \frac{a_{2i}}{a_{1i}} = q_{2i}^{-1}$ and our algebra satisfies (3.1.4.1c). We may rescale x'_1 and/or x'_2 (naming the new generators X_1 and X_2) such that R satisfies (3.1.4.1a) and thus all of (3.1.4.1). //

3.1.5 Remarks

- Conversely, any algebra of type (3.1.4.1) is a diffusion algebra. Simply start with a diffusion algebra where all the a_{ij}s are 1, all the b_{ij}s equal the intended q_{ij}, $r_1 = r_2 = 1$ and $r_i = 0$ for all $i \geqslant 3$.

- The algebras occurring in this proposition are higher generator analogues of the 2–generator possibilities when $q_{12} = q \neq 1$. These were the quantum affine plane and the quantised Weyl algebra. Our n–generator algebra turns out to be either multi-parameter quantum affine n–space (an n–generator analogue of the quantum plane) or to be an algebra with two distinguished indeterminates generating a quantised Weyl algebra and all the other indeterminates "q–commuting" in the most general way compatible with the quantised Weyl algebra's defining relation.

3.1.6 Proposition

If $q_{ij} \neq 0$ for all $i < j \in \{1, \ldots, n\}$ and $q_{ij} = 1$ for at least one pair $i < j \in \{1, \ldots, n\}$, then R is isomorphic either to multiparameter quantum affine n–space (possibly commutative polynomials in n indeterminates), to an algebra of type (3.1.4.1) *or to an algebra generated by indeterminates X_1, $x_2, \ldots, x_n$ subject to relations:*

$$X_1 x_i - x_i X_1 = k_i x_i \quad \text{for all } i \geqslant 2; \tag{3.1.6.1a}$$

$$x_i x_j - q_{ij} x_j x_i = 0 \quad \text{for all } j > i \geqslant 2, \tag{3.1.6.1b}$$

(3.1.6.1)

with $k_i \in \mathbb{C} \setminus \{0\}$ for all $i \geqslant 2$.

Proof:

We approach this initially by considering each of the 3–generator diffusion subalgebras generated by x_1, x_2 and x_i for $i \geqslant 3$ where, without loss of generality $q_{12} = 1$ (see (2.1.5)). By (3.1.2.1a) and (3.1.2.1c), we have $r_2 (q_{2i} - 1) = 0$ and $r_1 (q_{1i} - 1) = 0$ for all $i \geqslant 3$.

We work case by case:

(i) The case $r_1 = r_2 = 0$. The 3–generator subalgebras generated by x_1, x_2 and x_i for all $i \geqslant 3$ automatically satisfy all of (3.1.2.1) except for (3.1.2.1b) which states:

$$r_i \left(a_{1i}(q_{1i} - 1) - a_{2i}(q_{2i} - 1) \right) \text{ for all } i \geqslant 3.$$

If, for any $i \geqslant 3$, we have $q_{1i} = q_{2i} = 1$ and $r_i \neq 0$ then we exchange generators x_1 and x_i to obtain an n–generator diffusion algebra with $q_{12} = 1$ and $r_1 \neq 0$ (which will be dealt with in case (ii)). Otherwise, we may assume that $r_i = 0$ whenever $q_{1i} = q_{2i} = 1$. Moreover, by (3.1.2.1b), $r_i = 0$ whenever $q_{1i} \neq 1$ and $q_{2i} = 1$ and whenever $q_{1i} = 1$ and $q_{2i} \neq 1$. For any i for which $q_{1i} \neq 1 \neq q_{2i}$, we make the change of variable:

$$x_i' := x_i - \frac{r_i}{a_{1i}(1 - q_{1i})} \stackrel{(3.1.2.1b)}{=} x_i - \frac{r_i}{a_{2i}(1 - q_{2i})}.$$

Thus we may assume that all such 3–generator subalgebras satisfy:

$$x_1x_2 - x_2x_1 = 0; \tag{3.1.6.2a}$$

$$x_1x_i - q_{1i}x_ix_1 = 0 \text{ for all } i \geqslant 3; \tag{3.1.6.2b}$$

$$x_2x_i - q_{2i}x_ix_2 = 0 \text{ for all } i \geqslant 3. \tag{3.1.6.2c}$$

Considering the subalgebras generated by x_1, x_i and x_j for all $j > i \geqslant 3$, we have also:

$$x_ix_j - q_{ij}x_jx_i \in \{0, \kappa_{ij}\} \text{ for all } j > i \geqslant 3 \text{ where } \kappa_{ij} \in \mathbb{C} \setminus \{0\}. \tag{3.1.6.2d}$$

(3.1.6.2)

We consider $\left[x_i - \frac{r_i}{a_{1i}(1-q_{1i})}, x_j - \frac{r_j}{a_{1j}(1-q_{1j})}\right]_{q_{ij}}$ (where the symbol $[y,z]_q$ denotes the *q–commutator*, $yz - qzy$) as this encapsulates all possibilities of transformed and untransformed x_is (since in the untransformed case we always have $r_i = 0$):

$$\begin{aligned}\left[x_i - \frac{r_i}{a_{1i}(1-q_{1i})}, x_j - \frac{r_j}{a_{1j}(1-q_{1j})}\right]_{q_{ij}} =& r_j\left(\frac{1}{a_{ij}} - \frac{1-q_{ij}}{a_{1j}(1-q_{1j})}\right)x_i \\ & - r_i\left(\frac{1}{a_{ij}} + \frac{1-q_{ij}}{a_{1i}(1-q_{1i})}\right)x_j \\ & + \frac{r_ir_j(1-q_{ij})}{a_{1i}a_{1j}(1-q_{1i})(1-q_{1j})}.\end{aligned}$$

The coefficient of x_i is zero by (3.1.2.1b); that of x_j is zero by (3.1.2.1a). If $r_i = 0$ or $r_j = 0$ then the constant term is zero, otherwise neither of q_{1i} and q_{1j} are 1 and (3.1.2.1a) $\Rightarrow q_{ij} \neq 1$ and the constant term is $\kappa_{ij} \neq 0$. At most one pair $3 \leqslant i < j$ may have $x_ix_j - q_{ij}x_jx_i = \kappa_{ij} \neq 0$ otherwise, if r_k is also nonzero then we get a contradiction to lemma (3.1.3). Thus R is isomorphic either to multiparameter quantum affine n–space or an algebra of type (3.1.4.1).

(ii) The case $r_1 \neq 0$. In this case, we must, by (3.1.2.1b), have $q_{1i} = 1$ for all $i \geqslant 3$. We now identify two cases:

(a) $r_2 = 0$. In this case, we denote $x_i' := r_ix_1 - r_1x_i$ and we get:

$$x_2x_i' - q_{2i}x_i'x_2 \overset{(3.1.2.1\text{f})}{=} 0 \text{ for all } i \geqslant 3,$$

since whenever $r_i = 0$, this is just the defining relation, (2.1.1.1); whenever $r_i \neq 0$ we must, by (3.1.2.1b), have $q_{2i} = 1$.

To check the relations between x'_i and x'_j it is enough to consider the q–comutator, $[r_i x_1 - r_1 x_i, r_j x_1 - r_1 x_j]_{q_{ij}}$. If both r_i and r_j are zero, then this q_{ij}–commutator is zero simply from the defining relations, (2.1.1.1). Otherwise, note that if either r_i or r_j is nonzero then by (3.1.2.1a) or (3.1.2.1b) we have $q_{ij} = 1$ and thus:

$$\begin{aligned}[r_i x_1 - r_1 x_i, r_j x_1 - r_1 x_j]_{q_{ij}} = r_1^2 r_i r_j \Bigg(&\left(\frac{1}{a_{1i}} - \frac{1}{a_{1j}}\right) x_1 + \\ &\left(\frac{1}{a_{ij}} - \frac{1}{a_{1i}}\right) x_i + \\ &\left(\frac{1}{a_{1j}} - \frac{1}{a_{ij}}\right) x_j \Bigg).\end{aligned}$$

If either of r_i and r_j are zero then this is zero, otherwise (3.1.2.1d), (3.1.2.1e) and (3.1.2.1f) give us that $a_{1i} = a_{1j} = a_{ij}$ and this is again zero, as required.

(b) $r_2 \neq 0$. In this case, we must have $q_{2i} = 1$ by (3.1.2.1a) and $a_{1i} = a_{2i}$ by (3.1.2.1e). We make the change of variables $x'_2 := r_2 x_1 - r_1 x_2$. Whenever $r_i = 0$, we have $x'_2 x_i - x_i x'_2 = 0$. For any i for which $r_i \neq 0$ we make the change of variable $x'_i := r_i x_1 - r_1 x_i$. By (3.1.2.1d) and (3.1.2.1e) we have $a_{12} = a_{1i} = a_{2i}$. Then $a_{2i}\,(x'_2 x'_i - x'_i x'_2) = 0$. The calculation used in part (a) shows $[x_i, x_j]_{q_{ij}} = 0$, as required.

In either case we have the algebra defined by (3.1.6.1), as required.

(iii) The case $r_1 = 0$, $r_2 \neq 0$. We may exchange generators x_1 and x_2 to get into the preceding case.

//

3.1.7 Remarks

- Conversely, any algebra of type (3.1.6.1) is a diffusion algebra (in fact (3.1.6.1) is a *diffusion presentation*, i.e. a presentation of the form (2.1.1.1)).
- For algebras given by (3.1.6.1) we observe that, without loss of generality, one of the parameters k_i can be assumed to be 1.

- We again see the generalisation of the two-generator situation (this time when $q_{12} = 1$). In the 2–generator case we either got commutative polynomials in two variables or the universal enveloping algebra of the 2–d soluble lie algebra. In n generators, we either get quantum affine n–space (with at least one pair of commuting indeterminates) or an algebra with a single distinguished generator that obeys (a slight variation of) the "Lie algebra relation" with any other generator (all of which "q–commute"). For a more explicit description of this generalisation, see (3.2.8).

- If we allow at least one, but not all, of the parameters k_i to be zero in (3.1.6.1), our ring R will not be a diffusion algebra: we shall see in the next section that diffusion algebras of type (3.1.4.1) and (3.1.6.1) both have a unique non–maximal height $n-1$ prime. If $k_i = 0$, consider the factor ring of R modulo the ideal generated by all indeterminates x_j with $2 \leqslant j \neq i$. This is isomorphic to the subalgebra generated by X_1 and x_i, which is a commutative polynomial ring. It has infinitely many non–maximal height one primes which correspond to infinitely many non–maximal height $n-1$ primes of R. Moreover, R is not a multiparameter quantum affine n–space since its quotient division ring contains a Weyl algebra (generated by $k_\ell^{-1} X_1 x_\ell^{-1}$ and x_ℓ assuming $k_\ell \neq 0$). So it is not rationally equivalent to multiparameter quantum affine n–space since. In [Ric02, cor. 1.1.3.3], it was shown that the field of fractions of a multiparameter quantum affine n–space does not contain a Weyl algebra.

3.2 Prime Ideals

We shall assume that the reader is familiar with the fact that the prime spectrum of multiparameter quantum affine n–space generically (when the multiplicative group generated by the q_{ij}s is free abelian) consists of non–maximal ideals generated by any subset of upto $n-1$ of the n generators and maximal ideals generated by $n-1$ of the n generators and a scalar translate of the remaining one. We shall also assume that the reader is familiar with the fact that, nongenerically, the prime ideals of such an algebra lie in bijective correspondance with the spectra of the *quantum tori* obtained by factoring out some of the generators and localising at all the remaining ones and that, moreover, the primes of these rings lie in bijective correspondance with those of their centres which are generated by monomials (see [GL98]). We will examine the prime spectra of the two other types of noetherian diffusion algebra (3.1.4.1) and (3.1.6.1).

3.2.1 Lemma

Let R be an algebra as in (3.1.4.1) *and assume q_{12} is not a root of unity, then no proper ideal contains a power of X_1.*

Proof:

The subalgebra generated by X_1 and X_2 is a quantised Weyl algebra. Any proper ideal of R containing a power of X_1 contracts to an ideal of this quantised Weyl algebra. Such an ideal is trivial (see, for example, [Jor95, §2.6] which gives *skew commutator formulae* for powers of X_1 which allow us to reduce its degree until it is a nonzero scalar). //

3.2.2 Remarks

- As none of the parameters q_{ij} are allowed to be 1 in (3.1.4.1) the assumption that q_{12} is not a root of unity does not rule out any diffusion algebras that could genuinely arise from physics.
- Similarly, no proper ideal contains a power of X_2.
- It follows that the primes of R correspond bijectively (via the usual contraction and extension, see (1.5.3)) to those of the localisation at the denominator set of nonzero scalar multiples of powers of X_1 (by [Goo92, Lem 1.4], a right denominator set in a ring, S, extends to a right denominator set of a skew polynomial ring $S[x; \alpha, \delta]$ provided it is *alpha*–stable).

3.2.3 Proposition

The localisation, $R\mathbb{X}^{-1}$, of a diffusion algebra R as in (3.1.4.1) *where $\mathbb{X}$ is the denominator set of powers of X_1 can be viewed as the localisation, $\widetilde{R}\mathbb{X}^{-1}$, of a multiparameter quantum affine n–space $\widetilde{R}$ generated by X_1, C, $x_3, \ldots, x_n$.*

Proof:

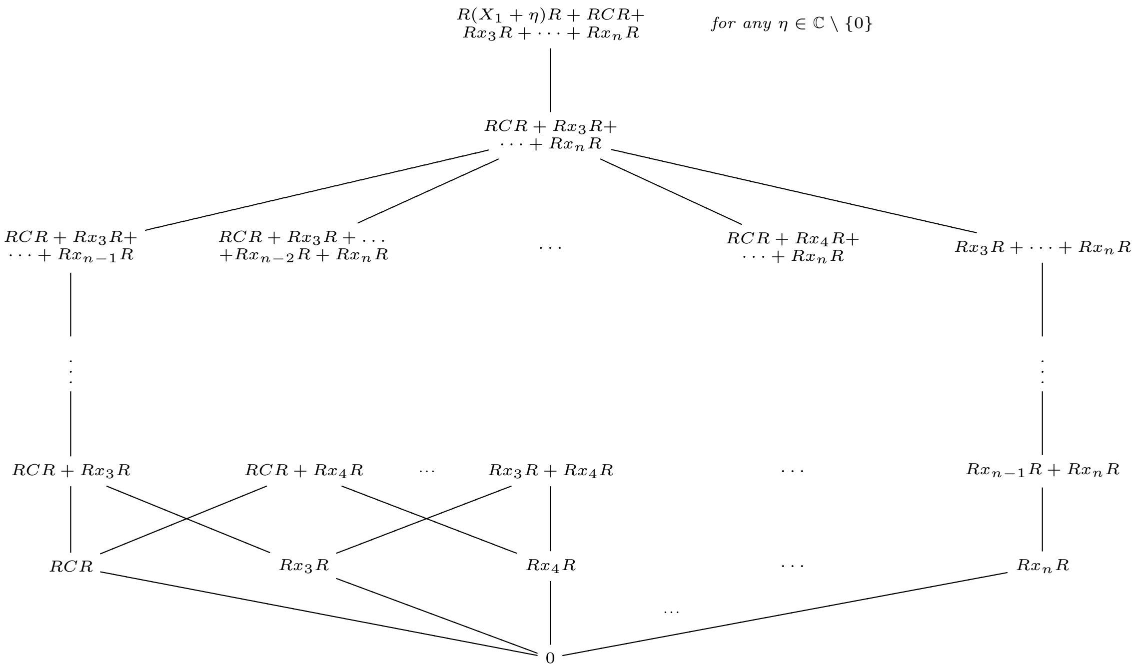
Let $C := (q-1)X_2X_1 + 1$ (the *casimir element* of the quantised Weyl subalgebra generated by X_1 and X_2). Then $X_2 = \frac{1}{q-1}(C-1)X_1^{-1}$ so may be replaced as a generator by C in $R\mathbb{X}^{-1}$. Moreover, $x_iC = Cx_i$ for all $i \geqslant 3$. The result follows. //

3.2.4 Remark

It follows that the prime ideals of R are in bijective correspondance with those primes of $\widetilde{R}$ not containing X_1. We will call a noetherian diffusion algebra of type (3.1.4.1) generic if and only if the multiplicative group generated by q_{12}, q_{1i} and q_{ij} for all $3 \leqslant i < j$ is free abelian.

3.2.5 Proposition

The prime spectrum of a generic diffusion algebra, R, of type (3.1.4.1) *looks like:*



i.e. The unique minimal prime is 0 *with principal ideals generated by each of* $C, x_3, \dots, x_n$ *at height one, ideals generated by each pair of elements from* $\{C, x_3, \dots, x_n\}$ *at height* 2 *and so on. At height* $n-1$ *we therefore have a unique prime generated by the whole set* $\{C, x_3, \dots, x_n\}$ *and we get infinitely many maximal ideals indexed by* $\mathbb{C} \setminus \{0\}$ *by adjoining* $X_1 + \eta$ *as an extra ideal generator.*

Proof:

We proceed by induction on the number of generators. The base cases $n = 1$ and $n = 2$ are easy (respectively, a polynomial ring in one indeterminate and a quantised Weyl algebra — for which 0, RCR and $RCR + R(X_1 + \eta)R$ where $\eta \neq 0$ are known to be all the primes). We now consider spec $\frac{R}{RCR}$ this is the spectrum of a generic quantum affine $n-1$–space localised at X_1, so the primes over RCR are as pictured. Consider also spec $\frac{R}{Rx_iR}$ for all $i \geqslant 3$, by induction, this is as pictured. It therefore suffices to prove that $RCR, Rx_3R, \ldots, Rx_nR$ are the only height one primes.

We may pass to the localisation of R by the set of all nonzero scalar multiples of monomials $X_1^{i_1}C^{i_2}x_3^{i_3}\ldots x_n^{i_n}$. This is a quantum torus, its primes are in bijective correspondence with those of its centre, which is trivial (i.e. just $\mathbb{C}$): by [GL98], the centre of the subalgebra generated by $X_1, x_3, \ldots, x_n$ and their inverses is trivial, a monomial involving a power of C cannot commute with X_1, $\mathbb{Z}^n$–grading this localisation by $\deg(X_1) := (1, 0, 0, 0, \ldots, 0), \deg(C) := (0, 1, 0, 0, \ldots, 0), \deg(x_3) := (0, 0, 1, 0, \ldots, 0), \ldots$ we deduce that nothing involving C commutes with X_1, thus the centre is trivial. //

The remainder of this section will be devoted to studying the prime spectrum of the less familiar algebra described in (3.1.6.1).

3.2.6 Lemma

Let R be an algebra given by (3.1.6.1)*, then the indeterminates, x_i are normal for all $i \geqslant 2$. An ideal generated by any number of these indeterminates is prime, as is the zero ideal.*

Proof:

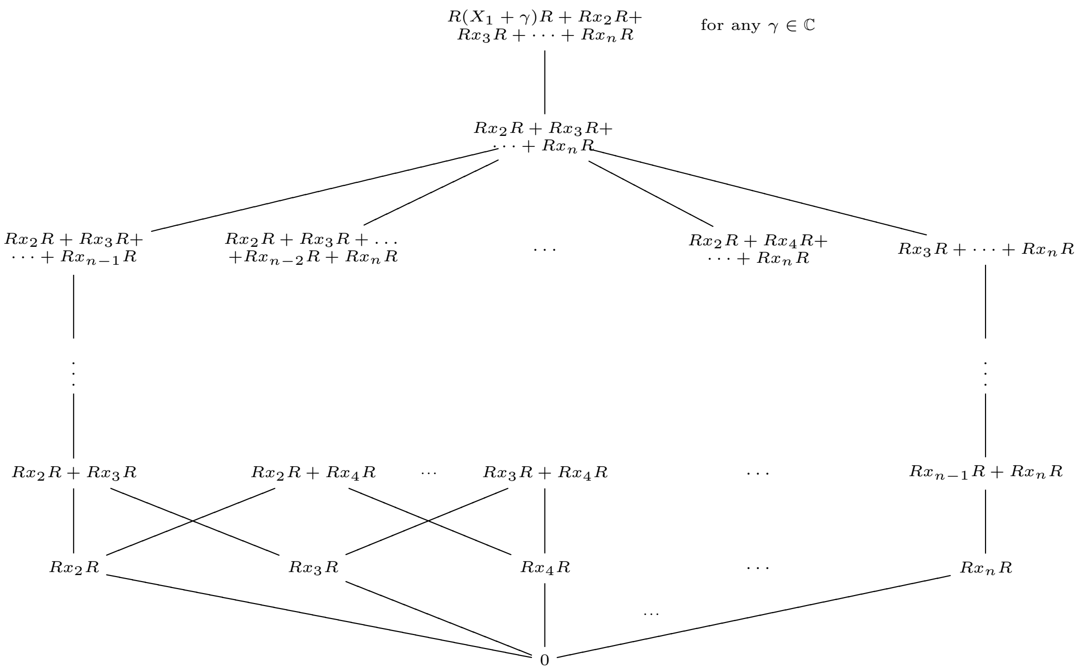
It is clear that the indeterminates x_i are normal for all $i \geqslant 2$. The algebras given by (3.1.6.1) are clearly domains and hence prime rings. The factor ring of R by the ideal generated by any number of these indeterminates x_i is another algebra given by (3.1.6.1) (including possibly the polynomial ring in one indeterminate, X_1) thus such an ideal is prime. //

3.2.7 Remarks

- There are also uncountably many maximal ideals generated by $X_1 + \gamma$, x_2, x_3, $\ldots$ and x_n where γ is any complex number.

- We noted in the proof, above, that the factor ring by a prime generated by some set of x_is ($i \geqslant 2$) is another algebra of type (3.1.6.1). We can analyse whether there are any more primes lying over this (other than those we have already discovered) by localising at all the other x_is. Thus it now suffices to understand the prime spectrum of the localisation of an algebra given by (3.1.6.1) at the denominator set generated by x_i for all $i \geqslant 2$.

- Generically (in a sense to be defined in (3.2.10)), we will see that we get no other prime ideals. Thus the prime spectrum looks like:



- A diffusion algebra of type (3.1.6.1) cannot be isomorphic to one of type (3.1.4.1). An algebra of type (3.1.6.1) has a unique non–maximal height $n-1$ prime (see (3.2.9) or the stratification by the group H in §6). A nongeneric algebra of type (3.1.4.1) can have more than one (in which case the rings are non-isomorphic). An algebra of type (3.1.4.1) has at least one such ideal, however, the ideal $RCR + Rx_3R + \cdots + Rx_nR$ if it is unique, the algebra is still not isomorphic to our algebra of type (3.1.6.1) since the factor modulo this ideal is a Laurent polynomial ring whereas the factor modulo the unique non–maximal height $n-1$ prime in an algebra of type (3.1.6.1) is an ordinary polynomial ring.

3.2.8 Lemma

Let S be the localisation of an algebra R given by (3.1.6.1) *at the denominator set generated by x_i for all $i \geqslant 2$, then S is isomorphic to the differential operator ring $T[X_1;\delta]$ where T is the subalgebra generated by $x_2^{\pm 1}, \ldots, x_n^{\pm 1}$, which is a quantum torus, and $\delta := \sum_{i=2}^{n} k_i x_i \frac{\partial}{\partial x_i}$.*

Proof:

Once we observe that (3.1.6.1) is actually a *diffusion presentation* (i.e. it is, provided some of the relations are divided by k_i, of the form (2.1.1.1)), this is easy. //

3.2.9 Proposition

With the notation of the preceeding lemma, $\mathbb{C}$–grade T by letting $\deg x_i := k_i$ *for all $i \geqslant 2$. Then contraction and extension provide a bijection between* $\operatorname{spec} S$ *and* $\operatorname{spec} \boldsymbol{\mathcal{Z}}(T)_0$*, the spectrum of the degree zero component of the centre.*

Proof:

First observe $\operatorname{spec} S$ corresponds bijectively to $\{\delta\text{–primes of } T\}$ by [GW89, thm 2.22(b)] (note that all primes of the base ring contain a normal — in fact central — element so we may drop the assumption that it need be commutative) since $x_2 \in \delta(T)$ and is invertible.

Next, observe that, both in T and $\boldsymbol{\mathcal{Z}}(T)$, δ–primes are prime δ–ideals by [Jor77, prop 1].

Observe also that, both in T and $\boldsymbol{\mathcal{Z}}(T)$, δ–ideals are graded ideals: Let m be any monomial in the indeterminates $x_2, \ldots, x_n$, then $\delta(m) = \deg(m)m$. Therefore if I is a

δ–ideal and $I \ni f = \sum_{\substack{\alpha \in \mathbb{C} \\ \text{finite sum}}} f_\alpha$ is a sum of homogeneous components each of degree α then $f_\alpha \in I$ for all α. Any graded ideal J can be generated by homogeneous elements and is therefore a δ–ideal.

By [GL98, cor 1.5] we have $\operatorname{spec} T \leftrightarrow \operatorname{spec} \mathcal{Z}(T)$ via contraction and extension. It is clear that the contraction of a graded ideal is again graded. Graded ideals extend to graded ideals since our graded ideal can be generated by homogeneous elements of degree zero (as, for any homogeneous element, we can multiply it by the inverse of a monomial of the same degree to get one of degree zero generating the same ideal) thus we have that $\{\text{prime graded ideals of } T\} \leftrightarrow \{\text{prime graded ideal of } \mathcal{Z}(T)\}$. Thus:

$$\begin{tikzcd}
\{\delta\text{–primes of } T\} \arrow[r, leftrightarrow] \arrow[d, equal] & \{\delta\text{–primes of } \mathcal{Z}(T)\} \arrow[d, equal] \\
\{\text{prime graded ideals of } T\} \arrow[r, leftrightarrow] & \{\text{prime graded ideals of } \mathcal{Z}(T)\}
\end{tikzcd}$$

Graded ideals of $\mathcal{Z}(T)$ correspond bijectively to ideals of the ring $\mathcal{Z}(T)_0$. Primes contract to primes in this correspondence since if $xy \in P \cap \mathcal{Z}(T)_0$ for $x, y \in \mathcal{Z}(T)_0$ we must have $xy \in P \Rightarrow x \in P$ or $y \in P \Rightarrow x \in P \cap \mathcal{Z}(T)_0$ or $y \in P \cap \mathcal{Z}(T)_0$. However, primes expand to gr–primes: Let I and J be graded ideals then they are generated by $I \cap \mathcal{Z}(T)_0$ and $J \cap \mathcal{Z}(T)_0$ thus, if $P \in \operatorname{spec} \mathcal{Z}(T)_0$, we have $IJ \subset P\,\mathcal{Z}(T) \Rightarrow (IJ) \cap \mathcal{Z}(T)_0 \subset P \Rightarrow (I \cap \mathcal{Z}(T)_0)(J \cap \mathcal{Z}(T)_0) \subset P \Rightarrow I \cap \mathcal{Z}(T)_0 \subset P$ or $J \cap \mathcal{Z}(T)_0 \subset P \Rightarrow I \subset P\,\mathcal{Z}(T)$ or $J \subset P\,\mathcal{Z}(T)$.

By [GL98] we have that $\mathcal{Z}(T) \cong \mathbb{C}[\underline{x}^{\pm \underline{n}_1}, \ldots, \underline{x}^{\pm \underline{n}_s}]$ for some s. Thus $\mathcal{Z}(T)$ is actually graded by the finitely generated additive subgroup, G, of $\mathbb{C}$ generated by $\deg(\underline{x}^{\underline{n}_1})$, $\deg(\underline{x}^{\underline{n}_2})$, $\ldots$, $\deg(\underline{x}^{\underline{n}_x})$ (thus $G \cong \frac{\mathbb{Z}^s}{H}$ for some subgroup H of G). G is torsion free so $G \cong \mathbb{Z}^r$ for some r.

By [BG02], for a $\mathbb{Z}^r$–grading, gr–prime ideals are prime. The result follows. //

3.2.10 Remarks

- $\mathcal{Z}(T)_0$ is generated as a $\mathbb{C}$–vector space by all monomials corresponding to elements of H (i.e. elements $\underline{x}^h$ for $h \in H$). However, H is free abelian since it is a subgroup of a free abelian group. Therefore $\mathcal{Z}(T)_0$ is a Laurent polynomial ring (in $s-r$ indeterminates).

- We shall call a diffusion algebra of type (3.1.6.1) generic if and only if $Z(T)_0$ is trivial. This happens precisely when the parameters k_i associated to any indeterminates x_i whose subscript is involved in a multiplicative relation amongst the q_{jk}s are linearly independent.

3.2.11 The Case $n = 3$

If we can identify the centre of T explicitly as a Laurent polynomial ring it is quite easy to explicitly identify the remaining primes (those nonzero primes not containing x_i for any $i \geqslant 2$). If $n = 3$, then we look at q_{23} if it is a non root of unity then $\mathcal{Z}(T) = \mathbb{C}$ (clearly, in this case there are no more primes except the zero ideal); if it is a primitive mth root of unity then $\mathcal{Z}(T) = \mathbb{C}[x_2^{\pm m}, x_3^{\pm m}]$. In this case, we look at k_2 and k_3. If $\frac{k_2}{k_3} \notin \mathbb{Q}$ then no monomial $m := x_3^{i_3}x_2^{i_2} \neq 1$ can have degree zero as $\deg m = i_3k_3 + i_2k_2$ (in this case, there are again no more primes except 0 as $\mathcal{Z}(T)_0 = \mathbb{C}$). If $i_3k_3 + i_2k_2 = 0$ with i_2 and i_3 coprime and nonzero and (without loss of generality) $i_2 \in \mathbb{N}$ then $\mathcal{Z}(T)_0 = \mathbb{C}[x_3^{i_3}x_2^{i_2}, x_3^{-i_3}x_2^{-i_2}]$ and for $i_3 > 0$ we get remaining primes 0 and maximals of the form $R(x_3^{i_3}x_2^{i_2} + \eta)R$ for all $\eta \in \mathbb{C} \setminus \{0\}$; for $i_3 < 0$ we get remaining primes 0 and maximals of the form $R(x_2^{i_2} + \eta x_3^{i_3})R$ for all $\eta \in \mathbb{C} \setminus \{0\}$.

3.3 Primitive Ideals

In this section we shall address the primitive ideals of the noetherian diffusion algebras. We shall, as usual, assume knowledge of the primitive spectra of quantum affine n–spaces (i.e. that, generically, all but the height $n-1$ primes are primitive; non–generically, precisely those primes corresponding to maximals of quantum tori when some indeterminates are factored out and others are localised are primitive, see [GL98]).

The height one prime spectra of the three different types of noetherian diffusion algebra are all quite similar. Generically, there are finitely many height one primes, each generated by a single normal element. In all other cases, there are uncountably many extra height one primes occurring in bijective correspondance with those of a nontrivial centre.

3.3.1 Definition

A noetherian diffusion algebra will be called height one generic if and only if it has finitely many height one prime ideals and these are each generated by normal elements $p_1, p_2, \ldots, p_n$, otherwise it will be called height one nongeneric.

Note that, in the situation of a noetherian algebra with the descending chain condition on prime ideals, this is the definition of a *unique factorisation ring* [CJ86] with the extra condition that the spectrum at height one should be finite.

3.3.2 Remarks

- Generic algebras are height one generic (as defined in §2 or as usual for multiparameter quantum affine n–space, i.e. the multiplicative group generated by the parameters, q_{ij} is free abelian).
- If R is a height one generic diffusion algebra with the p_i as given in the previous section then $\frac{R}{p_i R}$ is either another noetherian diffusion algebra or the localisation of a quantum affine n–space at one of the indeterminates. Thus, inductively, it suffices to address whether a height one generic diffusion algebra is a primitive ring or not.
- Note that all height one nongeneric diffusion algebras are nongeneric (again as defined in §2 or as usual for multiparameter quantum affine n–space) and thus have localisations with nontrivial centres. In the case of quantum affine n–spaces and diffusion algebras of type (3.1.4.1) we recall that the primes of a quantum torus are in bijective correspondence with its centre, see [GL98]. In the case of diffusion algebras of type (3.1.6.1) we observe that, with the notation of (3.2.9), the subalgebra $\mathcal{Z}(T)_0$ is central.

3.3.3 Lemma

Let R be any ring and $\mathbb{X}$ a multiplicative set of normal nonzerodivisors. If R is a primitive ring then so is $R\mathbb{X}^{-1}$.

Proof:

Let M_R be a simple module with $\operatorname{ann}_R(M) = 0$. Consider $\overline{M} = \frac{M}{T(M)} \hookrightarrow M\mathbb{X}^{-1}$. The set $\operatorname{ann}_M x := \{m \in M : mx = 0\}$ is a submodule of M for all $x \in \mathbb{X}$ since x is normal. Moreover, $\operatorname{ann}_M x \neq M$ since $x \notin \operatorname{ann}_R(M) = 0$ thus $\operatorname{ann}_M x = 0$ for all $x \in \mathbb{X}$ since M is simple. Hence $\operatorname{ass}(\mathbb{X}) = 0$ and $\overline{M} \cong M \subseteq M\mathbb{X}^{-1}$. Let $0 \neq N \leqslant M\mathbb{X}^{-1}$ then $N \cap M \neq 0$, thus $N \cap M = M$, thus $N = M\mathbb{X}^{-1}$, so $M\mathbb{X}^{-1}$ is simple. It is easy to see that we also have $\operatorname{ann}_{R\mathbb{X}^{-1}}(M\mathbb{X}^{-1}) = 0$, therefore $R\mathbb{X}^{-1}$ is primitive. //

3.3.4 Proposition

A noetherian diffusion algebra is a primitive ring if and only if it is height one generic.

Proof:

If R is a height one generic noetherian diffusion algebra then let I be a maximal right ideal containing the nonunit (this can be seen by considering degree) $p_1p_2\ldots p_n + 1$. The annihilator $\operatorname{ann}\left(\frac{R}{I}\right)$ of the simple right module $\frac{R}{I}$ is primitive therefore prime. If $\operatorname{ann}\left(\frac{R}{I}\right) \neq 0$ then $p_i \in \operatorname{ann}(\frac{R}{I})$ for some i, thus $p_1p_2\ldots p_n \in \operatorname{ann}\left(\frac{R}{I}\right)$, therefore $p_1p_2\ldots p_n \in I$ (since it is normal) a contradiction as this would mean I is trivial. Thus $\operatorname{ann}\left(\frac{R}{I}\right) = 0$.

Now let T be a localisation of a height one nongeneric noetherian diffusion algebra (specifically, the localisation, S, given in (3.2.8) for an algebra of type (3.1.6.1), a quantum torus for multiparameter quantum affine n–space and the localisation at monomials of the form $C^{i_2}x_3^{i_3}\ldots x_n^{i_n}$ for an algebra of type (3.1.4.1)) and assume, for a contradiction, that T is primitive. Let M_T be a simple module with $\operatorname{ann}(M) = 0$. Then $\boldsymbol{\mathcal{Z}}(T) \hookrightarrow \operatorname{End}(M)$ a contradiction since $\boldsymbol{\mathcal{Z}}(T)$ is infinite dimensional over $\mathbb{C}$ (for an algebra of type (3.1.6.1) we found part of the centre in (3.2.9), for a multiparameter quantum affine n–space this is well known and in the case of an algebra of type (3.1.4.1), consider the further localisation at powers of X_1, this is a quantum torus and any central monomial or its inverse was a central element in T as we simply need to ensure that X_1 occurs with positive degree) whereas, by [MR87, Thm 5.5], End(M) is finite dimensional since T is constructible (as defined in [MR87, §9.4.12], since it is an iterated skew polynomial ring). Thus, by the preceding lemma, a height one nongeneric diffusion algebra is not a primitive ring. //

3.3.5 Remarks

- If R is a generic noetherian diffusion algebra on $n \geqslant 1$ generators then all its prime ideals are primitive except for the unique height $n-1$ prime. The factor ring by any prime of height strictly less than $n-2$ (whenever such primes exist) is a height one generic diffusion algebra (or, sometimes, a localisation of a generic quantum affine m–space at one of the indeterminates, if R was of type (3.1.4.1)) which is primitive; the factor ring by the unique height $n-1$ prime is either a commutative polynomial ring or a commutative Laurent polynomial ring, neither of which is primitive; the height n primes are all maximal thus primitive.

- If R is a nongeneric noetherian diffusion algebra then its prime spectrum can be viewed as the union of several subposets one of which is isomorphic to the prime spectrum of a generic noetherian diffusion algebra of the same type. The others are isomorphic to the prime spectra of Laurent polynomial rings; these posets each consist of a prime of height not exceeding $n-2$ from the first poset and all the primes over it not contained in the first poset. The primitivity of the primes of height less than $n-2$ in the first subposet can be assessed using the preceding proposition since the factor rings by each are also noetherian diffusion algebras (or, sometimes, a localisation of a quantum affine m–space at one of the indeterminates, if R was of type (3.1.4.1)) which can be identified as either height one generic or height one nongeneric hence either primitive or not. The unique height $n-1$ prime in the first subposet is not primitive since the factor ring is a commutative polynomial (or Laurent polynomial) ring; the primes of height n are maximal therefore primitive. Among those primes in the other subposets, only the maximals are primitive as all the others give factor algebras with infinite dimensional centres.

3.4 Stratification

We will first let the algebraic torus $H := (\mathbb{C}^\star)^{n-1}$ act on a noetherian diffusion algebra, R, of type (3.1.6.1) via:

$$(\alpha_2, \alpha_3, \dots, \alpha_n) : x_i \longrightarrow \alpha_i x_i \text{ for all } i \geqslant 2;$$
$$(\alpha_2, \alpha_3, \dots, \alpha_n) : X_1 \longrightarrow X_1.$$

3.4.1 Proposition

With H acting on R as above, the H–primes of R are precisely the ideals generated by any subset of $\{x_2, x_3, \dots, x_n\}$ and the ideals of the form $R(X_1+\gamma)R+Rx_2R+\cdots+Rx_nR$ for any $\gamma \in \mathbb{C}$.

Proof:

The ideals are all H–ideals and we proved earlier (see (3.2.6) and (3.2.7)) that they are all prime thus they are H–primes. By [BG02, Ex II.1.J], any other H–primes lie in bijective correspondence (via localisation and contraction) with those of a localisation, S, isomorphic to the one found in (3.2.8), obtained by factoring out some of the x_is (but, of course, not all of them — these maximal ideals are already counted in the statement of the proposition) and localising at all the others. Let I be any non–zero H–ideal of S then $I \cap \mathbb{C}[X_1] \neq \{0\}$ (as we may pick out any non–zero H–homogeneous component and then multiply it by the inverses of any powers of the x_is occurring as factors). Let $f(X_1) \in (I \cap \mathbb{C}[X_1]) \setminus \{0\}$, if $\deg(f) = n > 0$ then $\deg(x_2 f x_2^{-1} - f) = n - 1$. To see this it suffices to prove, by induction, that $x_2 X_1^n x_2^{-1} = (X_1^n + nk_2 X_1^{n-1} + \dots)$. One can easily show similar formulae using x_i in place of x_2. The case $n = 1$ is easy.

$$\begin{aligned} x_2 X_1^n x_2^{-1} &= (X_1 + k_2) x_2 X_1^{n-1} x_2^{-1} \\ &= (X_1 + k_2)(X_1^{n-1} + (n-1)k_2 X_1^{n-2} + \dots) \qquad \text{(by induction)} \\ &= X_1^n + nk_2 X_1^{n-1} + \dots \end{aligned}$$

As $k_2 \neq 0$, $I \cap \mathbb{C} \neq \{0\}$ and therefore $I = S$ and S is H–simple. //

3.4.2 Corollary

We have $\operatorname{spec} R = \bigsqcup_{J \text{ an } H\text{–prime}} \operatorname{spec}_J R$ *and each of the strata,* $\operatorname{spec}_J R$*, is isomorphic to the prime spectrum of a commutative Laurent polynomial ring in at most* $n-1$ *variables.*

Proof:

Follows from (1.7.2). ⫽

Unfortunately, there are infinitely many H–primes with this choice of H. This is an inconvenience when trying to classify the primitive ideals (as we would like to use (1.7.3)). To rectify this we let the algebraic group $G := (\mathbb{C}, +) \times \big((\mathbb{C}^\star)^{n-1}, \times\big)$ act on R via:

$$(\alpha_1, \alpha_2, \dots, \alpha_n) : x_i \longrightarrow \alpha_i x_i \text{ for all } i \geqslant 2;$$
$$(\alpha_1, \alpha_2, \dots, \alpha_n) : X_1 \longrightarrow X_1 + \alpha_1.$$

3.4.3 Proposition

With G *acting on* R *as above, the* G*–primes are precisely the ideals generated by any subset of* $\{x_2, x_3, \dots, x_n\}$.

Proof:

We prove that G–prime ideals are also H–prime. Let P be G–prime and consider a prime, Q, minimal over P. The H–prime $(Q : H) := \bigcap_{h \in H} h(Q)$ satisfies $P \subseteq (Q : H) \subseteq Q$ but, by [BG02, Prop II.2.9], $(Q : H)$ is prime hence $(Q : H) = Q$. By [BG02, Lem II.1.10] we have $P = (Q : G)$ for any prime Q (now known, necessarily, to also be an H–prime) minimal over P. There are finitely many primes (all, necessarily, H–primes) minimal over P since R is noetherian but from our classification of the H–primes we see that their G–orbits are either singletons (for non–maximal H–primes) or uncountable (for maximal H–primes, a contradiction). Thus $P = Q$ is a non–maximal H–prime. ⫽

3.4.4 Corollary

We have $\operatorname{prim} R = \bigsqcup_{J \text{ a } G\text{–prime}} \{$*maximal elements of* $\operatorname{spec}_J R\}$ *where the strata,* $\operatorname{spec}_J R$*, are as before except that* $\operatorname{spec}_{Rx_2R+\cdots+Rx_nR} R = \{Rx_2R+\cdots+Rx_nR\} \cup \{R(X_1+\gamma)R + Rx_2R + \cdots + Rx_nR :$ *for all* $\gamma \in \mathbb{C}\}$.

Proof:

Follows from (1.7.3). //

Finally, we will let H act on a diffusion algebra, R, of type (3.1.4.1) via:

$$(\alpha_2, \alpha_3, \ldots, \alpha_n) : x_i \longrightarrow \alpha_i x_i \text{ for all } i \geqslant 3;$$
$$(\alpha_2, \alpha_3, \ldots, \alpha_n) : X_1 \longrightarrow \alpha_2^{-1} X_1;$$
$$(\alpha_2, \alpha_3, \ldots, \alpha_n) : X_2 \longrightarrow \alpha_2 X_2.$$

3.4.5 Proposition

With H *acting on* R *as above and* q_{12} *not a root of unity, the* H*–primes are precisely those ideals generated by any subset of* $\{C, x_3, x_4, \ldots, x_n\}$.

Proof:

These ideals are all H–stable primes, thus are H–primes. The maximals, $RCR + R(X_1+\eta)R+Rx_3R+\cdots+Rx_nR$, are not H–stable thus any other H–primes lie in bijective correspondance (Again, via localisation and contraction, as in [BG02, Ex II.1.J]) with those of a localisation obtained by factoring out some of the elements of $\{C, x_3, x_4, \ldots, x_n\}$ and localising at powers of all the others. If we localised at powers of C then we get an H–simple ring as any H–ideal intersects the subalgebra generated by X_1 and X_2 (take any H homogeneous element of the ideal and premultiply by the x_is as necessary to be left with an element of this subalgebra) thus contains a power of C which is a now unit; if we factored out C then our localisation is a quantum torus which is H–simple, by [BG02, Ex. II.2.K]. //

3.4.6 Corollary

We have $\operatorname{spec} R = \bigsqcup_{J \text{ an } H\text{–prime}} \operatorname{spec}_J R$ *and*

$$\operatorname{prim} R = \bigsqcup_{J \text{ a } G\text{–prime}} \{\text{maximal elements of } \operatorname{spec}_J R\}$$

where each of the strata, $\operatorname{spec}_J R$*, is isomorphic to the prime spectrum of a commutative Laurent polynomial ring in at most* $n-1$ *variables.*

Proof:

Follows from (1.7.2) and (1.7.3). //

3.4.7 Remark

Noetherian diffusion algebras have *normal separation* (a spectrum is normally seperated if, whenever $P \subset Q$ for P and Q prime, $\frac{Q}{P}$ contains a normal element of $\frac{R}{P}$). For multiparameter quantum affine n–space, this was proved in [GL96, cor 2.4]. For algebras of type (3.1.4.1) and type (3.1.6.1) we can apply [BG02, thm II.9.15(d)]. Since, in view of (2.2.6), noetherian diffusion algebras are Cohen-Macauley and Auslander-regular, we can conclude that noetherian diffusion algebras are catenary and Tauvel's height formula holds, see [GL96, thm 1.6].

3.4.8 Remark

We can easily see that the actions of H are rational from [BG02, II.2.6] by observing that the characters are the same as those for the action on a multiparameter quantum affine $n-1$–space.

We can see that the action of G is rational by considering the product morphisms $\mathbb{C} \times H \hookrightarrow \mathrm{GL}(V_i)$ for a suitable choice of V_i.

Chapter 4

Non–Noetherian Diffusion Algebras

4.1 Classification

We will now examine 3 generator diffusion algebras. We will assume that $q_{12} = 0$ and will partition the other diffusion algebras by looking at the three different possibilities for the other two q_{ij}s (i.e. that they are either 0, 1 or any other complex number). In the next two propositions, we will identify the various different types of algebra that will occur. These types will be summarised — and further classified — in the theorem that follows.

4.1.1 Proposition

Let R be a 3–generator diffusion algebra with $q_{12} = 0$ and $q_{i3} = 1$ for at least one $i \in \{1, 2\}$, then either R or R^{op} is isomorphic to one of the following algebras:

(i) *The algebra generated by X_1, X_2 and X_3 subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 - q_{13}X_3X_1 = 0;$$
$$X_2X_3 - q_{23}X_3X_2 = 0.$$

(ii) *The algebra generated by X_1, X_2 and X_3 subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 - qX_3X_1 = pX_1 \quad \textit{where } p \in \mathbb{C} \setminus \{0\};$$
$$X_2X_3 - X_3X_2 = X_2.$$

(iii) *The algebra generated by X_1, X_2 and X_3 subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 - X_3X_1 = \lambda X_1 + \mu X_3 + \kappa \quad \textit{where } \lambda, \mu, \kappa \in \mathbb{C},\ \lambda \textit{ and } \mu \textit{ not both zero;}$$
$$X_2X_3 = 0.$$

Proof:

Observe that, if $q_{23} \neq 0 \neq q_{13}$ then we may replace R with R^{op} and reorder $x_1' := x_2^{\mathrm{op}}$, $x_2' := x_1^{\mathrm{op}}$, $x_3' := x_3^{\mathrm{op}}$ to effectively interchange q_{13} and q_{23} using (2.1.5) and (2.1.7). Thus, provided $q_{23} \neq 0$, we may assume $q_{13} = q \in \mathbb{C}$ and $q_{23} = 1$. Under this assumption (2.1.4.1) becomes:

$$qr_2 = 0; \tag{4.1.1.1a}$$
$$r_1\left(a_{12} - a_{13}(1-q)\right) = 0; \tag{4.1.1.1c}$$
$$r_2r_3\left(a_{13} - a_{12}\right) = 0; \tag{4.1.1.1d}$$
$$r_1r_2\left(a_{13} - a_{12} - a_{23}\right) = 0; \tag{4.1.1.1e}$$
$$r_1r_3 = 0. \tag{4.1.1.1f}$$

(4.1.1.1)

By (4.1.1.1a), either $q = 0$ or $r_2 = 0$. Assume (for now) the latter then, by (4.1.1.1f), either $r_1 = 0$ or $r_3 = 0$. If $r_3 \neq 0$ then letting $X_3 := \frac{a_{23}}{r_3} x_3$ and $p := \frac{a_{23}}{a_{13}}$ we recognise (4.1.1(ii)). If $r_3 = 0$ then let $X_1 := x_1 + \frac{r_1}{(1-q)a_{13}} \overset{(4.1.1.1c)}{=} x_1 + \frac{r_1}{a_{12}}$ to see that our algebra fits into class (4.1.1(i)).

We now look at what happens if $q = 0$, we may assume $r_2 \neq 0$. (4.1.1.1c) and (4.1.1.1d) give us that either $a_{12} = a_{13}$ or $r_1 = r_3 = 0$. In the former case, (4.1.1.1e) give us that $r_1 = 0$, let $X_1 := x_1$, $X_2 := \frac{r_3 x_2 - r_2 x_3}{a_{23}}$ and $X_3 := \frac{a_{23} x_2}{r_2}$ to recognise (4.1.1(ii)) with $p = \frac{a_{23}}{a_{13}}$. In the latter case, we let $X_1 := x_1$, $X_2 := x_3$ and $X_3 := -\frac{a_{23} x_2}{r_2}$ and recognise (4.1.1(ii)) with $p = -\frac{a_{23}}{a_{12}}$.

It only remains to consider the case $q_{23} = 0$, it must therefore be q_{13} that is 1. Thus (2.1.4.1) becomes:

$$r_2 \left(a_{23} + a_{12}\right) = 0; \tag{4.1.1.2a}$$

$$r_2 r_3 \left(a_{13} - a_{23} - a_{12}\right) = 0; \tag{4.1.1.2d}$$

$$r_1 r_2 \left(a_{13} - a_{23} - a_{12}\right) = 0. \tag{4.1.1.2e}$$

(4.1.1.2)

If $r_2 \neq 0$, we must have $a_{12} = -a_{23}$ therefore we must also have $r_1 = r_3 = 0$. Letting $X_2 := x_2 - \frac{r_2}{a_1 2} = x_2 + \frac{r_2}{a_2 3}$ we recognise (4.1.1(i)). Whereas, if $r_2 = 0$, we let $X_1 := x_1 + \frac{r_1}{a_{12}}$, $X_2 := x_2$ and $X_3 := x_3 - \frac{r_3}{a_{23}}$ and recognise (4.1.1(iii)) with $\lambda = \frac{r_3}{a_{13}}$, $\mu = -\frac{r_1}{a_{13}}$ and $\kappa = -\frac{r_1 r_3 (a_{23} + a_{12})}{a_{12} a_{13} a_{23}}$ (unless, in addition $r_1 = r_2 = 0$ in which case we have something of the type (4.1.1(i))).

//

4.1.2 Proposition

Let R be a 3–generator diffusion algebra with $q_{12} = 0$ and $q_{i3} \in \mathbb{C} \setminus \{1\}$ then R is isomorphic to one of the following algebras:

(i) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$X_1X_2 - q_{12}X_2X_1 = 0;$$
$$X_1X_3 - q_{13}X_3X_1 = 0;$$
$$X_2X_3 - q_{23}X_3X_2 = 0,$$

observe that this is (4.1.1(i)) *again.*

(iv) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$a_{12}X_1X_2 = r_2X_1 - r_1X_2;$$
$$a_{13}X_1X_3 = r_3X_1 - r_1X_3;$$
$$a_{23}X_2X_3 = r_3X_2 - r_2X_3,$$

where the parameters satisfy (2.1.4.1d) *and* (2.1.4.1e).

(v) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 - qX_3X_1 = \lambda X_1 + \mu X_3 + \kappa;$$
$$X_2X_3 = 0,$$

where $\lambda, \mu, \kappa \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0, 1\}$, λ *and* μ *not both zero.*

(vi) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 = 0;$$
$$X_2X_3 - qX_3X_2 = \lambda X_2 + \mu X_3$$

where $\lambda, \mu \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0, 1\}$, λ *and* μ *not both zero.*

(vii) *The algebra generated by X_1, X_2 and X_3 subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 - q_1X_3X_1 = X_1;$$
$$X_2X_3 - q_2X_3X_2 = 0 \quad \textit{where } q_1, q_2 \in \mathbb{C} \setminus \{0, 1\}.$$

(viii) *The algebra generated by X_1, X_2 and X_3 subject to relations:*

$$X_1X_2 = 1;$$
$$X_1X_3 - qX_3X_1 = 0;$$
$$X_2X_3 - q^{-1}X_3X_2 = 0 \quad \textit{where } q \in \mathbb{C} \setminus \{0, 1\}.$$

Proof:

With $q_{12} = 0$, (2.1.4.1) becomes:

$$q_{13}r_2\left(a_{23}(q_{23} - 1) - a_{12}\right) = 0; \tag{4.1.2.1a}$$
$$q_{23}r_1\left(a_{13}(q_{13} - 1) + a_{12}\right) = 0; \tag{4.1.2.1c}$$
$$r_2r_3\left(a_{23}(q_{23} - 1) + a_{13} - a_{12}\right) = 0; \tag{4.1.2.1d}$$
$$r_1r_2\left(a_{13} - a_{12} - a_{23}\right) = 0; \tag{4.1.2.1e}$$
$$r_1r_3q_{23} = 0. \tag{4.1.2.1f}$$

(4.1.2.1)

Observe, first of all, that (4.1.2.1f) give us that either $q_{23} = 0$, $r_1 = 0$ or $r_3 = 0$.

Assuming $q_{23} = 0$ then (4.1.2.1a) gives us either

(a) that $q_{13} = 0$, in which case we have an algebra as in (4.1.2(iv)).

(b) that $r_2 = 0$, in which case letting $X_1 := x_1 + \frac{r_1}{a_{12}}$, $X_2 := x_2$ and $X_3 := x_3 - \frac{r_3}{a_{23}}$ we recognise (4.1.2(v)). Note that, in this case, we get: $\lambda := r_3\left(\frac{1}{a_{13}} - \frac{1-q_{13}}{a_{23}}\right)$, $\mu := -r_1\left(\frac{1}{a_{13}} - \frac{1-q_{13}}{a_{12}}\right)$, $\kappa := -\frac{r_1r_3}{a_{12}a_{13}a_{23}}\left(a_{12} + a_{23} - (1 - q_{13})a_{13}\right)$. In particular, $\frac{\kappa}{\lambda\mu} \neq \frac{-1}{1-q_{13}}$ (whenever it makes sense, $\frac{\kappa}{\lambda\mu} = \frac{-1}{1-q_{13}} \Leftrightarrow a_{12}a_{23} = 0$, a contradiction).

or (c) that $a_{23} = -a_{12} \overset{(4.1.2.1d)\&(4.1.2.1e)}{\Longrightarrow} r_1 = r_3 = 0$ (since we may assume $r_2 \neq 0$), in which case letting $X_2 := x_2 - \frac{r_2}{a_{12}} = x_2 + \frac{r_2}{a_{23}}$ we recognise (4.1.1(i)).

We may now assume $q_{23} \neq 0$ and $r_1 = 0$ then (4.1.2.1a) gives us either

(a) that $q_{13} = 0$, in which case letting $X_1 := x_1$, $X_2 := x_2 - \frac{r_2}{a_{12}}$ and $X_3 := x_3 - \frac{r_3}{a_{13}}$ we recognise (4.1.2(vi)) where, in the final relation, $\lambda := r_3\left(\frac{1}{a_{23}} - \frac{1-q_{23}}{a_{13}}\right)$, $\mu := -r_2\left(\frac{1}{a_{23}} + \frac{1-q_{23}}{a_{12}}\right)$ and (4.1.2.1d) tells us that the constant term is zero.

or (b) that $q_{13} \neq 0$ in which case (4.1.2.1a) and (4.1.2.1d) tell us that either $r_2 = 0$ or $r_3 = 0$. If $r_3 = 0$ then letting $X_2 := x_2 - \frac{r_2}{a_{12}} \overset{(4.1.2.1a)}{=} x_2 + \frac{r_2}{a_{23}(1-q_{23})}$ we recognise (4.1.1(i)) otherwise we let $X_3 := \frac{a_{23}(1-q_{23})x_3}{r_3(a_{23}(1-q_{23})-a_{13}(1-q_{13}))} - \frac{1}{(a_{23}(1-q_{23})-a_{13}(1-q_{13}))}$ we recognise (4.1.2(vii)).

Finally, we assume $q_{23} \neq 0$ and $r_1 \neq 0$ then $r_3 = 0$. Either

(a) we have $r_2 = 0$, in which case letting $X_1 := x_1 + \frac{r_1}{a_{12}} \overset{(4.1.2.1c)}{=} x_1 + \frac{r_1}{a_{13}(1-q_{13})}$ gives us an algebra of type (4.1.1(i)).

or (b) we have $r_2 \neq 0$ in which case (4.1.2.1c)&(4.1.2.1e) $\Rightarrow q_{13} \neq 0$. Let $X_1 := \frac{a_{12}x_1}{r_1} + 1 \overset{(4.1.2.1c)}{=} -\frac{a_{13}(q_{13}-1)x_1}{r_1} + 1$, $X_2 := -\frac{a_{12}x_2}{r_2} + 1 \overset{(4.1.2.1a)}{=} -\frac{a_{23}(q_{23}-1)x_2}{r_2} + 1$ and $X_3 := x_3$ to recognise (4.1.2(viii)). Note that (4.1.2.1a)&(4.1.2.1e) $\Rightarrow q_{23} = \frac{a_{13}}{a_{23}}$ and (4.1.2.1c)&(4.1.2.1e) $\Rightarrow q_{13} = \frac{a_{23}}{a_{13}}$ thus q_{13} and q_{23} are mutually inverse, as required.

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The following theorem summarises the previous two propositions whilst subclassifying the various types of algebra to improve on the classification and to prepare for the analysis of the prime ideals.

4.1.3 Theorem

Every non–noetherian 3–generator diffusion algebra is isomorphic to one of the following types of algebra:

(I) *The algebras given by presentations of the form:*

$$X_1X_2 = 0;$$
$$X_1X_3 - q_1X_3X_1 = 0;$$
$$X_2X_3 - q_2X_3X_2 = 0.$$

These are:

(a) *The case* $q_1 = q_2 = 0$.

(b) *The case* $q_1 = 0$, $q_2 = 1$.

(c) *The case* $q_1 = 0$, $q_2 \in \mathbb{C} \setminus \{0\}$ *not a root of unity.*

(d) *The case* $q_1 = 1$, $q_2 = 0$.

(e) *The case* $q_1 \in \mathbb{C} \setminus \{0\}$ *not a root of unity,* $q_2 = 0$.

(f) *The case* $q_1 = 1$, $q_2 = 1$.

(g) *The case* $q_1 = 1$, $q_2 \in \mathbb{C} \setminus \{0\}$ *not a root of unity.*

(h) *The case* $q_1, q_2 \in \mathbb{C} \setminus \{0\}$ *not roots of unity.*

(II) *The algebras given by presentations of the form:*

$$X_1X_2 = 0;$$
$$X_1X_3 - qX_3X_1 = pX_1;$$
$$X_2X_3 - X_3X_2 = X_2,$$

where $p \in \mathbb{C} \setminus \{0\}$. *These are:*

(a) *The case* $q = 0$.

(b) *The case* $q = 1$.

(c) *The case* $q \in \mathbb{C} \setminus \{0, 1\}$ *not a root of unity.*

(III) *The algebras given by presentations of the form:*

$$X_1X_2 = 0;$$
$$X_1X_3 - qX_3X_1 = \lambda X_1 + \mu X_3 + \kappa;$$
$$X_2X_3 = 0.$$

These are:

(a) *The case* $q = 1$, $\lambda = \mu = 1$, $\kappa \in \mathbb{C} \setminus (\{0\} \cup -\mathbb{N})$.

(b) *The case* $q = 1$, $\lambda = \mu = 1$, $\kappa \in -\mathbb{N}$.

(c) *The case* $q \in \mathbb{C} \setminus \{0, 1\}$ *not a root of unity,* $\lambda = \mu = 1$, $\kappa \neq 0, \frac{-1}{1-q}$ *and* $[n]_q + \kappa \neq 0$ *for all* $n \in \mathbb{N}$.

(d) *The case* $q \in \mathbb{C} \setminus \{0, 1\}$ *not a root of unity,* $\lambda = \mu = 1$ *and* $[n]_q + \kappa = 0$*for somen* $\in \mathbb{N}$.

(e) *The case* $q = 1$, $\lambda = \mu = 1$, $\kappa = 0$.

(f) *The case* $q \in \mathbb{C} \setminus \{0, 1\}$ *not a root of unity,* $\lambda = \mu = 1$, $\kappa = 0$.

(g) *The case* $q \in \mathbb{C} \setminus \{0, 1\}$ *not a root of unity,* $\lambda = 1$, $\mu = \kappa = 0$.

(IV) *The algebras given by presentations of the form:*

$$a_{12}X_1X_2 = r_2X_1 - r_1X_2;$$
$$a_{13}X_1X_3 = r_3X_1 - r_1X_3;$$
$$a_{23}X_2X_3 = r_3X_2 - r_2X_3,$$

where the r_i *are not all zero. These are:*

(a) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$
$$X_1X_3 = 0;$$
$$(X_2 + 1)X_3 = 0.$$

(b) *The algebra generated by* X_1*,* X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$

$$X_1(X_3 + 1) = 0;$$

$$X_2X_3 = 0.$$

(c) *The algebra generated by* X_1*,* X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$

$$X_1X_3 = 1;$$

$$X_2X_3 = X_2.$$

(d) *The algebra generated by* X_1*,* X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$

$$X_1X_3 = 1;$$

$$X_2X_3 = 0.$$

(e) *The algebra generated by* X_1*,* X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = X_2;$$

$$X_1X_3 = \kappa;$$

$$X_2X_3 = X_2,$$

where $\kappa \in \mathbb{C} \setminus \{0, 1\}$.

(f) *The algebra generated by* X_1*,* X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = 0;$$

$$X_1X_3 = X_1 + X_3;$$

$$X_2X_3 = 0,$$

where $\lambda \in \mathbb{C} \setminus \{0,1\}$.

(g) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = X_1 + 1;$$
$$X_1X_3 = \frac{1}{1+\lambda}\left(X_1 - X_3\right);$$
$$X_2X_3 = -\lambda X_3 + 1,$$

where $\lambda \in \mathbb{C} \setminus \{0,1\}$.

(h) *The algebra generated by* X_1, X_2 *and* X_3 *subject to relations:*

$$X_1X_2 = X_1 + X_2;$$
$$X_1X_3 = 0;$$
$$X_2X_3 = 0.$$

(V) *The algebras given by presentations of the form:*

$$X_1X_2 = 0;$$
$$X_1X_3 = 0;$$
$$X_2X_3 - qX_3X_2 = \lambda X_2 + \mu X_3.$$

These are:

(a) *The case* $q \in \mathbb{C} \setminus \{0,1\}$ *not a root of unity,* $\lambda = 1$, $\mu = 0$.

(b) *The case* $q \in \mathbb{C} \setminus \{0,1\}$ *not a root of unity,* $\lambda = \mu = 1$.

(VI) *The algebra given by the presentation:*

$$X_1X_2 = 0;$$
$$X_1X_3 - q_1X_3X_1 = X_3;$$
$$X_2X_3 - q_2X_3X_2 = 0,$$

where $q_1, q_2 \in \mathbb{C} \setminus \{0,1\}$ *not roots of unity.*

(VII) *The algebra given by the presentation:*

$$X_1X_2 = 1;$$
$$X_1X_3 - qX_3X_1 = 0;$$
$$X_2X_3 - q^{-1}X_3X_2 = 0,$$

where $q \in \mathbb{C} \setminus \{0, 1\}$ *not a root of unity.*

Proof:

This is mostly easy. Type I is just (4.1.1(i)) where we have observed that, without loss of generality, $q_{12} = 0$ and we do not need to list the possibility $q_1 \in \mathbb{C} \setminus \{0\}$ not a root of unity, $q_2 = 1$ as this is the opposite ring of type I(g) with the change of PBW–basis order putting X_3^{op} first (again using (2.1.5) and (2.1.7)). Type II is just (4.1.1(ii)). Type III encompasses (4.1.1(iii)) and (4.1.2(v)), observing that, without loss of generality, $\lambda, \mu \in \{0, 1\}$.

Type IV requires most work. By (2.1.4.1d) and (2.1.4.1e) we have:

$$r_2r_3(a_{13} - a_{12} - a_{23}) = 0;$$
$$r_1r_2(a_{13} - a_{12} - a_{23}) = 0.$$

We first assume $r_1 = r_3 = 0$ and make the change of variables $X_2' := a_{12}X_2 - r_2$. This is either type I or we let $X_2'' := \frac{a_{23}}{r_2(a_{12}+a_{23})}X_2'$ to get type IV(a). Secondly, we assume $r_2 = 0$ and, without loss of generality, $r_1 \neq 0$. We make the change of variables $X_1' := a_{12}X_1 + r_1$, $X_3' := a_{23}X_3 - r_3$, thus:

$$X_1'X_2 = 0;$$
$$a_{13}X_1'X_3' = r_3(a_{23} - a_{13})X_1' + r_1(a_{13} - a_{12})X_3' + r_1r_3(a_{13} - a_{12} - a_{23});$$
$$X_2X_3' = 0.$$

If $r_3 = 0$ then we may rescale X_1' to get type IV(b). We may now assume $r_3 \neq 0$ then, without loss of generality, $r_1 = r_3 = 1$, assume $a_{13} = a_{12}$ then either $a_{23} = a_{13}$, in which case we have IV(d) or $a_{23} \neq a_{13}$, in which case we let $X_3'' := \frac{-X_3'}{a_{23}-a_{13}} + 1$, $X_1'' := \frac{(a_{23}=a_{13})X_1'}{a_{23}}$ to get type IV(c). We may now assume $r_1 = r + 3 = 1$, $a_{13} \neq a_{12}$ and, without loss of generality, $a_{23} \neq a_{13}$, if $a_{13} - a_{12} - a_{23} = 0$, we may rescale X_1' and X_3' to get IV(f), otherwise we let $X_1''' := \frac{a_{13}a_{12}}{a_{12}-a_{13}} X_1 + \frac{a_{12}}{a_{12}-a_{13}}$, $X_3''' := \frac{a_{13}a_{23}}{a_{13}-a_{23}} X_3 - \frac{a_{23}}{a_{13}-a_{23}}$ to get type IV(e) with $\kappa = -\frac{a_{12}a_{23}}{(a_{12}-a_{13})(a_{13}-a_{23})} \in \mathbb{C}\backslash\{0,1\}$. Thirdly, and finally, we assume $a_{13} - a_{12} - a_{23} = 0$, without loss of generality$r_1 = r_2 = 1$. If $r_3 = 0$ then let $X_1' := \frac{a_{13}}{a_{12}-a_{13}} X_1 + \frac{1}{a_{12}-a_{13}}$, $X_2' := \frac{a_{23}}{a_{12}+a_{23}} X_2 + \frac{1}{a_{12}+a_{23}}$ to get type IV(h). Otherwise, without loss of generality $r_3 = 1$, letting $X_1' := -a_{12}X_1 - 1$, $X_2' := a_{12}X_2$, $X_3' := -\frac{a_{23}^2}{a_{12}} X_3 + \frac{a_{23}}{a_{12}}$ we get type IV(g) with $\lambda = \frac{a_{12}}{a_{23}} \in \mathbb{C} \setminus \{0,1\}$.

Type V is just (4.1.2(vi)) with the observation that, without loss of generality $\lambda, \mu \in \{0,1\}$ and since we may exchange X_2 and X_3 (again by (2.1.5)) we may assume $\lambda = 1$. Types VI and VII are just (4.1.2(vii)) and (4.1.2(viii)), respectively. //

4.2 Prime Ideals

Observe that, up to isomorphism, there are precisely two 2–generator non–noetherian diffusion algebras:

- The algebra generated by X_1 and X_2 subject to the relation $X_1X_2 = 0$.
- The algebra generated by X_1 and X_2 subject to the relation $X_1X_2 = 1$.

This can easily be seen by letting $X_1 := a_{12}x_1 + r_1$ and $X_2 := a_{12}x_2 - r_2$ to get the algebra generated by X_1 and X_2 subject to the relation $X_1X_2 = r_1r_2$ and observing that, without loss of generality, $r_1r_2 \in \{0,1\}$. We will consider their prime spectra to be understood (the first is easily understood — $(RX_1R)(RX_2R) = RX_1X_2R = 0$ thus any prime contains either X_1 or X_2; the second was studied in [Jor00] — it is a prime ring with a unique height one prime, $R(X_1X_2 - 1)R$, whose factor ring is a Laurent polynomial ring in one variable), we will also assume the reader is familiar with the prime spectra of 2–generator noetherian diffusion algebras (the quantum affine plane, the quantised Weyl algebra and the enveloping algebra of the 2–d non–abelian soluble Lie algebra).

We examine first the prime spectra of a type I diffusion algebra, R.

4.2.1 Lemma

Any prime ideal in a type I *diffusion algebra contains either X_1 or X_2.*

Proof:

Consider the product of ideals $(RX_1R)(RX_2R)$. An element of this product is a $\mathbb{C}$–linear combination of monomials of the form:

$$X_3^{i_3}X_2^{i_2}X_1^{i_1}X_3^{j_3}X_2^{j_2}X_1^{j_1}$$

where $j_1, j_3, i_2, i_3 \in \mathbb{Z}^+$, $i_1, j_2 \in \mathbb{N}$.

$$X_3^{i_3}X_2^{i_2}X_1^{i_1}X_3^{j_3}X_2^{j_2}X_1^{j_1} = q_{13}^{i_1 j_3}X_3^{i_3}X_2^{i_2}X_3^{j_3}X_1^{i_1}X_2^{j_2}X_1^{j_1} = 0$$

since $i_1, j_2 \geqslant 1$ and $X_1X_2 = 0$. Therefore $(RX_1R)(RX_2R) = 0$, thus every prime ideal contains either X_1 or X_2. //

4.2.2 Proposition

The prime spectra of the type I *diffusion algebras are as pictured:*

- *Type* I(a)*:*

$$\begin{tikzcd}
RX_1R+RX_2R+R(X_3+\gamma)R & \cdots & R(X_1+\gamma)R+RX_2R+RX_3R & \cdots & RX_1R+R(X_2+\gamma)RX_3R \\
RX_1R+RX_2R \arrow[u, dash] & & RX_2R+RX_3R \arrow[u, dash] & & RX_1R+RX_3R \arrow[u, dash]
\end{tikzcd}$$

where $\gamma \in \mathbb{C}$.

- *Type* I(b)*:*

$$\begin{tikzcd}
RX_1R+R(X_2+\gamma_1)R+R(X_3+\gamma_2)R & \cdots & R(X_1+\gamma)R+RX_2R+RX_3R \\
RX_1R+Rp(X_2,X_3)R \arrow[u, dash] & & \\
RX_1R \arrow[u, dash] & & RX_2R+RX_3R \arrow[uu, dash]
\end{tikzcd}$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{C}$ and $p \in \mathbb{C}[X_2, X_3]$ is an irreducible polynomial.

- *Type* I(c)*:*

$$\begin{tikzcd}
RX_1R+RX_2R+R(X_3+\gamma)R & \cdots & RX_1R+R(X_2+\gamma)R+RX_3R & \cdots & R(X_1+\gamma)R+RX_2R+RX_3R \\
RX_1R+RX_2R \arrow[u, dash] & & RX_1R+RX_3R \arrow[u, dash] & & \\
& RX_1R \arrow[ul, dash] \arrow[ur, dash] & & & RX_2R+RX_3R \arrow[uu, dash]
\end{tikzcd}$$

where $\gamma \in \mathbb{C}$*.*

- *Type* I(d)*:*

$$\begin{tikzcd}
RX_1R + R(X_2+\gamma)R + RX_3R \arrow[dd, dash] & \cdots & R(X_1+\gamma_1)R + RX_2R + R(X_3+\gamma_2)R \arrow[d, dash] \\
 & & RX_2R + Rp(X_1,X_3)R \arrow[d, dash] \\
RX_1R + RX_3R & & RX_2R
\end{tikzcd}$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{C}$ *and* $p \in \mathbb{C}[X_1, X_3]$ *is an irreducible polynomial.*

- *Type* I(e)*:*

$$\begin{tikzcd}
RX_1R + R(X_2+\gamma)R + RX_3R \arrow[dd, dash] & \cdots & RX_1R + RX_2R + R(X_3+\gamma)R \arrow[d, dash] & \cdots & R(X_1+\gamma)R + RX_2R + RX_3R \arrow[d, dash] \\
 & & RX_1R + RX_2R \arrow[dr, dash] & & RX_2R + RX_3R \arrow[dl, dash] \\
RX_1R + RX_3R & & & RX_2R &
\end{tikzcd}$$

where $\gamma \in \mathbb{C}$*.*

- *Type* I(f)*:*

$$\begin{tikzcd}
RX_1R + R(X_2+\gamma_1)R + R(X_3+\gamma_2)R \arrow[d, dash] & \cdots & R(X_1+\gamma_1)R + RX_2R + R(X_3+\gamma_2)R \arrow[d, dash] \\
RX_1R + Rp(X_2,X_3)R \arrow[d, dash] & \cdots & RX_2R + Rq(X_1,X_3)R \arrow[d, dash] \\
RX_1R & & RX_2R
\end{tikzcd}$$

where $\gamma_1, \gamma_2 \in \mathbb{C}$*,* $p \in \mathbb{C}[X_2, X_3]$ *and* $q \in \mathbb{C}[X_1, X_3]$ *are irreducible polynomials.*

- *Type* I(g)*:*

$$\begin{tikzcd}
\begin{matrix} RX_1R + RX_3R \\ +R(X_2+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} RX_1R + RX_2R \\ +R(X_3+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} R(X_1+\gamma_1)R + RX_2R \\ +R(X_3+\gamma_2)R \end{matrix} \arrow[d, dash] \\
RX_1R + RX_3R \arrow[dr, dash] & & RX_1R + RX_2R \arrow[dl, dash] & \cdots & RX_2R + Rp(X_1,X_3)R \arrow[d, dash] \\
 & RX_1R & & & RX_2R
\end{tikzcd}$$

where $\gamma, \gamma_1\gamma_2 \in \mathbb{C}$ *and* $p \in \mathbb{C}[X_1, X_3]$ *is an irreducible polynomial.*

- *Type* I(h)*:*

$$\begin{tikzcd}
\begin{matrix} RX_1R + RX_3R \\ +R(X_2+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} RX_1R + RX_2R \\ +R(X_3+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} R(X_1+\gamma)R+ \\ RX_2R + RX_3R \end{matrix} \arrow[d, dash] \\
RX_1R + RX_3R \arrow[dr, dash] & & RX_1R + RX_2R \arrow[dl, dash] \arrow[dr, dash] & & RX_2R + RX_3R \arrow[dl, dash] \\
 & RX_1R & & RX_2R &
\end{tikzcd}$$

where $\gamma \in \mathbb{C}$*.*

Proof:

By the preceding lemma, R is not prime and its prime spectra may be studied via the factor rings $\frac{R}{RX_1R}$, which is isomorphic to the subalgebra generated by X_2 and X_3, and $\frac{R}{RX_2R}$, which is isomorphic to the subalgebra generated by X_1 and X_3. These are known cases. //

We now examine the prime spectra when R is a type II diffusion algebra.

4.2.3 Lemma

Every prime ideal of a type II *diffusion algebra contains either X_1 or X_2.*

Proof:

Consider the product of ideals $(RX_1R)(RX_2R)$. An element of this product is a $\mathbb{C}$–linear combination of monomials of the form:

$$X_3^{i_3}X_2^{i_2}X_1^{i_1}X_3^{j_3}X_2^{j_2}X_1^{j_1}$$

where $i_2, i_3, j_1, j_3 \in \mathbb{Z}^+$, $i_1, j_2 \in \mathbb{N}$.

$$X_3^{i_3}X_2^{i_2}X_1^{i_1}X_3^{j_3}X_2^{j_2}X_1^{j_1} = X_3^{i_3}X_2^{i_2}X_1^{i_1-1}(qX_3+p)^{j_3}X_1X_2^{j_2}X_1^{j_1} = 0$$

since $i_1, j_2 \geqslant 1$ and $X_1X_2 = 0$. Therefore $(RX_1R)(RX_2R) = 0$, thus every prime contains either X_1 or X_2.

4.2.4 Proposition

The prime spectra of the type II *diffusion algebras are as pictured:*

- *Type* II(a)*:*

$$\begin{tikzcd}
RX_1R+RX_2R+R(X_3+\gamma)R \arrow[d, dash] & \cdots & R(X_1+\gamma)R+RX_2R+R(X_3-p)R \arrow[dd, dash] \\
RX_1R+RX_2R \arrow[d, dash] & & \\
RX_1R & & RX_2R+R(X_3-p)R
\end{tikzcd}$$

where $\gamma \in \mathbb{C}$.

- *Type* II(b)*:*

$$\begin{tikzcd}[column sep=small]
 & RX_1R + RX_2R + R(X_3+\gamma)R \arrow[d, dash] & \\
 & RX_1R + RX_2R \arrow[dl, dash] \arrow[dr, dash] & \\
RX_1R & & RX_2R
\end{tikzcd}$$

  where $\gamma \in \mathbb{C}$*.*

- *Type* II(c)*:*

$$\begin{tikzcd}[column sep=small]
RX_1R + RX_2R + R(X_3+\gamma)R \arrow[d, dash] & \cdots\ R(X_1+\gamma)R + RX_2R + R(X_3 - \frac{p}{1-q})R \arrow[d, dash] \\
RX_1R + RX_2R \arrow[d, dash] \arrow[dr, dash] & RX_2R + R(X_3 - \frac{p}{1-q})R \arrow[d, dash] \\
RX_1R & RX_2R
\end{tikzcd}$$

  where $\gamma \in \mathbb{C}$*.*

Proof:

By the preceding lemma, we can analyse the prime spectra via the factor rings $\frac{R}{RX_1R}$, which is isomorphic to the subalgebra generated by X_2 and X_3, and $\frac{R}{RX_2R}$, where:

$$\frac{R}{RX_2R} \cong \begin{cases} \mathbb{C}\left\langle X_1, (X_3 - p) : X_1(X_3 - p) = 0 \right\rangle & \\ & \text{for type II(a);} \\ \mathbb{C}\left\langle X_1, \left(\frac{X_3}{p}\right) : X_1\left(\frac{X_3}{p}\right) - \left(\frac{X_3}{p}\right) X_1 = X_1 \right\rangle & \\ & \text{for type II(b);} \\ \mathbb{C}\left\langle X_1, \left(X_3 - \frac{p}{1-q}\right) : X_1\left(X_3 - \frac{p}{1-q}\right) - q\left(X_3 - \frac{p}{1-q}\right) X_1 = 0 \right\rangle & \\ & \text{for type II(c).} \end{cases}$$

These are all known cases. //

We now move on to the prime spectra of type III diffusion algebras.

4.2.5 Lemma

Let R be a type III diffusion algebra then:

- *We have* $X_2RX_2 = X_2^2\mathbb{C}[X_2]$*.*

- *The ideal RX_2R is prime.*
- *The factor rings $\frac{R}{RX_2R}$ are as follows:*
  - *Types III(a),III(b)&III(e): the universal enveloping algebra of the 2–d soluble Lie algebra,*
  - *Types III(c),III(d)&III(f): the quantised Weyl algebra,*
  - *Type III(g): the quantum affine plane.*

Proof:

The first part is easy, the second follows from the third. If R is of one of the types III(a), III(b) or III(e), then

$$\frac{R}{RX_2R} \cong \mathbb{C}\left\langle \left(X_1+X_3+\kappa\right), X_3 : \left(X_1+X_3+\kappa\right)X_3 - X_3\left(X_1+X_3+\kappa\right) = \left(X_1+X_3+\kappa\right)\right\rangle,$$

which is the universal enveloping algebra of the 2–d soluble Lie algebra. If R is of one of the types III(c), III(d) or III(f), then

$$\frac{R}{RX_2R} \cong \mathbb{C}\left\langle \left(X_1-\frac{1}{1-q}\right), \left(X_3-\frac{1}{1-q}\right) : \left(X_1-\frac{1}{1-q}\right)\left(X_3-\frac{1}{1-q}\right) - q\left(X_3-\frac{1}{1-q}\right)\left(X_1-\frac{1}{1-q}\right) = \frac{1}{1-q}+\kappa\right\rangle,$$

which is the quantised Weyl algebra (since $\kappa \neq \frac{-1}{1-q}$). Finally, if R is of type III(g), then $\frac{R}{RX_2R} \cong \mathbb{C}\left\langle X_1, \left(X_3-\frac{1}{1-q}\right) : X_1\left(X_3-\frac{1}{1-q}\right) - q\left(X_3-\frac{1}{1-q}\right)X_1 = 0\right\rangle$, which is the quantum affine plane. //

4.2.6 Lemma

Let R be a type III(e), III(f) or III(g) diffusion algebra, then $RX_1R + RX_3R = RX_1 + X_3R$. Moreover, this and RX_2R are precisely the minimal primes.

Proof:

For the first part it suffices to observe that the reductions given by the defining relations respect the property of containing either X_1 or X_3 in all monomials and that the PBW–basis has powers of X_3 on the left and powers of X_1 on the right. For the second part, we see that $RX_2R(RX_1R + RX_3R)RX_2R = RX_2(RX_1R + RX_3R)X_2R = RX_2(RX_1 + X_3R)X_2R = RX_2RX_1X_2R + RX_2X_1RX_2R = 0$. //

4.2.7 Proposition

The prime spectra of type III(e), III(f) and III(g) diffusion algebras are as follows:

- *Type III(e):*

$$\begin{tikzcd}
RX_1R + R(X_2+\gamma)R + RX_3R \arrow[dd, dash] & \cdots & R(X_1+X_3)R + RX_2R + R(X_3+\gamma)R \arrow[d, dash] & \text{for any } \gamma \in \mathbb{C} \\
 & & R(X_1+X_3)R + RX_2R \arrow[d, dash] & \\
RX_1R + RX_3R & & RX_2R &
\end{tikzcd}$$

- *Type III(f):*

$$\begin{tikzcd}
RX_1R + R(X_2+\gamma)R + RX_3R \arrow[dd, dash] & \cdots & \begin{array}{c} RX_2R + R((q-1)X_3X_1 + X_3 + X_1)R \\ +R(X_3 - \frac{1}{1-q} + \eta)R \end{array} \arrow[d, dash] & \text{for any } \gamma \in \mathbb{C},\ \eta \in \mathbb{C} \setminus \{0\} \\
 & & R((q-1)X_3X_1 + X_3 + X_1)R + RX_2R \arrow[d, dash] & \\
RX_1R + RX_3R & & RX_2R &
\end{tikzcd}$$

- *Type III(g):*

$$\begin{tikzcd}
\begin{array}{c} RX_1R + R(X_2+\gamma)R \\ +RX_3R \end{array} \arrow[dd, dash] & \cdots & \begin{array}{c} RX_1R + RX_2R \\ +R(X_3+\gamma)R \end{array} \arrow[d, dash] & \cdots & \begin{array}{c} R(X_1+\gamma)R + RX_2R \\ +R(X_3 - \frac{1}{1-q})R \end{array} \arrow[d, dash] & \text{for any } \gamma \in \mathbb{C} \\
 & & RX_1R + RX_2R \arrow[dr, dash] & & RX_2R + R(X_3 - \frac{1}{1-q})R \arrow[dl, dash] & \\
RX_1R + RX_3R & & & RX_2R & &
\end{tikzcd}$$

Proof:

This follows from the preceding lemmas, the observation that $\frac{R}{RX_1R+RX_3R} \cong \mathbb{C}[X_2]$ and the observation that, in type III(f), the quantised Weyl algebra $\frac{R}{RX_2R}$ has *Casimir* element $(q-1)X_3X_1 + X_3 + X_1$. //

4.2.8 Lemma

If an ideal of a quantised Weyl algebra $\mathbb{C}\langle x, z : xz - qzx = 1\rangle$ ($q \neq 0$ not a root of unity) contains a non–zero element f of total degree N then it also contains C^N (where $C := xz - zx = (q-1)zx + 1$ denotes the casimir element).

Proof:

Recall that $xC = qCx$, $Cz = qzC$ and denote $ad_{q^j}(x)(—) := [x,—]_{q^j}$. Observe that $\left(ad_{q^{i-1}}(x)\right)^i(z^i) = \alpha C^i +$ (terms of lower degree in z) for some non–zero α. This is since $xz^j = q^j z^j x +$ (terms of lower degree in z) thus $ad_{q^{i-1}}(x)(z^i) = q^{i-1}(z^{i-1}xz - z^i x) +$ (terms of lower degree in z) $= q^{i-1}z^{i-1}C +$ (terms of lower degree in z). Using $Cx = q^{-1}xC$ we can iterate $(i-1)$ more times to get $\left(ad_{q^{i-1}}(x)\right)^i(z^i) = q^{\frac{i(i-1)}{2}}C^i +$ (terms of lower degree in z) and q is not a root of unity.

Let $f = \sum\limits_{i=0}^{n} z^i f_i(x)$ with $f_n \neq 0$ and $\mathrm{totdeg}(f) = N$ be an element of our ideal. Applying each in turn of the following operators (if necessary) to f: $ad_{q^{n-1}}(x)^n$; $ad_{q^{n-2}}(x)^{n-1}$; $\ldots$; $ad_q(x)^2$; $ad(x)$, we may replace f with an element $f' = \sum\limits_{i=0}^{n} C^i f_i'(x)$, also in our ideal, where we may assume without loss of generality that $f_n'(x) = x^k f_n(x)$ (as we can divide by non–zero scalars). Observing that $ad_{q^i}(x)(—)$ kills precisely the C^i term of f' we may replace it with $f'' = C^n f_n''(x)$, another element of our ideal, where we may again assume without loss of generality that $f_n''(x) = x^{k'} f_n(x)$.

Now, $\deg(f_n) + n \leqslant \mathrm{totdeg}(f) = N$ and $[—, C]_{q^i}$ kills precisely the x^i term of $f_n(x)$ (let $\deg(f_n) := m$) thus we may conclude that $C^{n+m}x^{k''}$ is in our ideal for some $k'' \in \mathbb{N}$ now using the skew commutator formulae [Jor95, §2.6] we may conclude that C^{n+m} is in our ideal and $n + m \leqslant N$ thus C^N is in our ideal. //

4.2.9 Lemma

Let R be a diffusion algebra of type III(a), III(b), III(c) or III(d) and let $P(X_3) \in \mathbb{C}[X_3]$ be a polynomial of degree n (not to be confused with the n used to describe κ in the definition of III(d)), then:

- *We have $(X_1 - [n]_q)P(X_3)X_2 = P'(X_3)X_2$ for some $P'(X_3) \in \mathbb{C}[X_3]$ of degree at* *most $n-1$.*

- *Moreover, there is a unique (upto scalar multiplication) polynomial $P(X_3)$ of degree n such that $(X_1 - [n]_q)P(X_3)X_2 = 0$.*

- *Lastly $(X_1 - [m]_q)P(X_3)X_2 = P''(X_3)X_2$ for some $P''(X_3) \in \mathbb{C}[X_3]$ of degree at most n for any $m \in \mathbb{Z}^+$.*

Proof:

The third part is clear. To prove the first two parts, we first establish that $X_1X_3^nX_2 = \sum_{i=0}^{n-1}(qX_3+1)^i(X_3+\kappa)X_3^{n-1-i}X_2$ by induction on n. The $n=1$ case is easily verified: $X_1X_3X_2 = (qX_3X_1 + X_1 + X_3 + \kappa)X_2 = (X_3+\kappa)X_2$. Now $X_1X_3^nX_2 = (qX_3X_1 + X_1 + X_3 + \kappa)X_3^{n-1}X_2 = (qX_3+1)X_1X_3^{n-1}X_2 + (X_3+\kappa)X_3^{n-1}X_2$, the result then follows from the induction hypothesis.

Without loss of generality, $P(X_3)$ is monic, i.e. $P(X_3) = X_3^n + p_{n-1}X_3^{n-1} + \cdots + p_1X_3 + p_0$, then:

$$\begin{aligned}
(X_1 - [n]_q)P(X_3)X_2 =& \sum_{i=0}^{n-1}(qX_3+1)^i(X_3+\kappa)X_3^{n-1-i}X_2 \\
&+ p_{n-1}\sum_{i=0}^{n-2}(qX_3+1)^i(X_3+\kappa)X_3^{n-2-i}X_2 \\
&+ \cdots + p_1(X_3+\kappa)X_2 \\
&- [n]_q(X_3^n + p_{n-1}X_3^{n-1} + \cdots + p_1X_3 + p_0)X_2 \\
=&(X_3+\kappa)\left(\sum_{i=0}^{n-1}\sum_{j=0}^{i}\begin{pmatrix} i \\ j \end{pmatrix} q^{i-j}X_3^{n-1-j}\right. \\
&\left.+p_{n-1}\sum_{i=0}^{n-2}\sum_{j=0}^{i}\begin{pmatrix} i \\ j \end{pmatrix} q^{i-j}X_3^{n-2-j} + \cdots + p_1\right)X_2 \\
&- [n]_q(X_3^n + p_{n-1}X_3^{n-1} + \cdots + p_1X_3 + p_0)X_2
\end{aligned}$$

From this we see that the terms of degree n in X_3 cancel, hence the first part. Moreover, requiring that the coefficient of X_3^{n-1} is zero uniquely determines the value of p_{n-1}, having chosen this value of p_{n-1} the value of p_{n-2} is uniquely determined by requiring that the coefficient of X_3^{n-2} is zero, iteratively we obtain a unique monic polynomial, $P(X_3)$, of degree n such that $(X_1 - [n]_q)P(X_3)X_2 = 0$. //

4.2.10 Remarks

- We shall call the polynomials $P(X_3)$ such that we have $(X_1 - [n]_q)P(X_3)X_2 = 0$ the <u>q–bad polynomials</u>.
- There is a corresponding result on the other side for reducing the degree of polynomials $Q(X_1)$ appearing as $X_2Q(X_1)$ by postmultiplying by $(X_3 - [n]_q)$.

4.2.11 Lemma

If R is as in the previous lemma, the polynomial, $P_n(X_3) := \prod_{i=1}^{n}\left(X_3 + \frac{([i-1]_q+\kappa)}{q^{i-1}}\right)$, is the unique monic q–bad polynomial of degree n.

Proof:

After the previous lemma, it suffices to show that $(X_1 - [n]_q)P_n(X_3)X_2 = 0$ for all $n \in \mathbb{N}$. We proceed by induction, firstly $(X_1 - 1)(X_3 + \kappa)X_2 = 0$. Let $n > 1$ then

$$\begin{aligned}
&(X_1 - [n]_q)P_n(X_3)X_2 \\
&\qquad = (X_1 - [n]_q)\left(X_3 + \frac{[n-1]_q + \kappa}{q^{n-1}}\right) P_{n-1}(X_3)X_2 \\
&\qquad = (qX_3X_1 + X_1 + X_3 + \kappa)P_{n-1}(X_3)X_2 + \frac{[n-1]_q + \kappa}{q^{n-1}} X_1 P_{n-1}(X_3)X_2 \\
&\qquad\quad - [n]_q\left(X_3 + \frac{[n-1]_q + \kappa}{q^{n-1}}\right) P_{n-1}(X_3)X_2 \\
&\qquad = \left(qX_3 + 1 + \frac{[n-1]_q + \kappa}{q^{n-1}}\right) X_1 P_{n-1}(X_3)X_2 \\
&\qquad\quad + (X_3 + \kappa)P_{n-1}(X_3)X_2 - [n]_q\left(X_3 + \frac{[n-1]_q + \kappa}{q^{n-1}}\right) P_{n-1}(X_3)X_2 \\
&\qquad \overset{\text{ind. hyp.}}{=} \left(qX_3 + 1 + \frac{[n-1]_q + \kappa}{q^{n-1}}\right)[n-1]_q P_{n-1}(X_3)X_2 + (X_3 + \kappa)P_{n-1}(X_3)X_2 \\
&\qquad\quad - [n]_q\left(X_3 + \frac{[n-1]_q + \kappa}{q^{n-1}}\right) P_{n-1}(X_3)X_2 \\
&\qquad = \begin{pmatrix} (q[n-1]_q + 1)X_3 - [n]_qX_3 + [n-1]_q + \\ \kappa + ([n-1]_q - [n]_q)\left(\dfrac{[n-1]_q + \kappa}{q^{n-1}}\right) \end{pmatrix} P_{n-1}(X_3)X_2 \\
&\qquad = 0
\end{aligned}$$

Thus $P_n(X_3)$ is the unique monic q–bad polynomial of degree n. By symmetry $Q_n(X_1) := \prod_{i=1}^{n}\left(X_1 + \frac{[i-1]_q+\kappa}{q^{i-1}}\right)$ is the unique monic q–bad polynomial of degree n in X_1. //

4.2.12 Lemma

Let R be either a type III(a), III(b), III(c) or III(d) diffusion algebra, let $C := (q-1)X_3X_1 + X_1 + X_3 + \kappa$ (this is the Casimir element *of the quantised Weyl algebra $\mathbb{C}[X_3, X_1]$ for a type III(c) or III(d) diffusion algebra), then we have $X_2C^nX_2 = \left(\prod_{i=1}^{n}([i-1]_q + \kappa)\right)X_2^2$.*

Proof:

We proceed by induction. First observe $X_2CX_2 = \kappa X_2^2$. We note that

$$\begin{aligned} X_1C = X_1\left((q-1)X_3X_1 + X_1 + X_3 + \kappa\right) \\ = (q-1)(qX_3X_1 + X_1 + X_3 + \kappa) + X_1^2 + qX_3X_1 + X_1 + X_3 + \kappa + \kappa X_1 \\ = q\left((q-1)X_3X_1 + X_1 + X_3 + \kappa\right)X_1 + (q-1)X_3X_1 + X_1 + X_3 + \kappa \\ = C(qX_1 + 1). \end{aligned}$$

Now

$$\begin{aligned} X_2C^nX_2 = X_2((q-1)X_3X_1 + X_1 + X_3 + \kappa)C^{n-1}X_2 \\ = X_2(X_1 + \kappa)C^{n-1}X_2 \\ = \kappa X_2C^{n-1}X_2 + X_2C^{n-1}(q^{n-1}X_1 + [n-1]_q)X_2 \\ = ([n-1]_q + \kappa)X_2C^{n-1}X_2 \end{aligned}$$

the result then follows by induction. //

4.2.13 Proposition

Let R be a diffusion algebra of one of the types III(a) or III(c), and let I be a nonzero two–sided ideal of R, then $I \bigcap X_2\mathbb{C}[X_2] \neq \{0\}$.

Proof:

Let $f \in I\setminus\{0\}$, write $f = g+h$ where g is in the subalgebra generated by X_1 and X_3 and h is a $\mathbb{C}$–linear combination of monomials of the form $X_3^{i_3}X_2^{i_2}X_1^{i_1}$ where $i_2 \geqslant 1$. We may assume, without loss of generality, that $g = 0$ since if R is of type III(a), then the subalgebra generated by X_1 and X_3 is the universal enveloping algebra of the 2–d soluble lie algebra, $\mathbb{C}\left\langle (X_1 + X_3 + \kappa), X_3 : (X_1 + X_3 + \kappa)X_3 - X_3(X_1 + X_3 + \kappa) = (X_1 + X_3 + \kappa)\right\rangle$ every ideal of which contains a power of $(X_1 + X_3 + \kappa)$ thus we can write $(X_1 + X_3 + \kappa)^m = \sum\limits_i r_igr_i'$ for some $r_i, r_i' \in \mathbb{C}[X_1, X_3]$. Let $f' := \sum\limits_i r_ifr_i' = (X_1 + X_3 + \kappa)^m + h'$ where h' is still a $\mathbb{C}$–linear combination of monomials of the form $X_3^{i_3}X_2^{i_2}X_1^{i_1}$ where $i_2 \geqslant 1$. Let $f'' := X_2f'X_2 = \kappa^mX_2^2 +$ terms of higher X_2 degree, replacing f with f'' yields the claim that without loss of generality $g = 0$. If R is of type III(c), we can use the same technique to replace f with $f' := C^m + h'$ (since every ideal of the quantised Weyl algebra $\mathbb{C}[X_1, X_3]$ contains a power of C). Replacing f' with f'' and applying (4.2.12) yields again the claim that without loss of generality $g = 0$.

Write $f = \sum_{\substack{i_2 \in \mathbb{N} \\ i_1 \in \mathbb{Z}^+ \\ \text{finite sum}}} f_{i_2 i_3}(X_3) X_2^{i_2} X_1^{i_1}$ where $f_{i_2 i_3}(X_3) \in \mathbb{C}[X_3]$. Observe that, after (4.2.11), all the q–bad polynomials have non-zero constant terms. If any of the polynomials $f_{i_2 i_3}(X_3)$ have nonzero constant terms, $f_{i_2 i_3 0}$, then replace f with $X_2 f = \sum_{i_2 i_3} f_{i_2 i_3 0} X_2^{i_2+1} X_1^{i_1} \neq 0$. Otherwise we may deduce that the polynomial amongst the $f_{i_2 i_3}(X_3)$s of maximal degree (n, say) is not a q–bad polynomial. By (4.2.9), we may replace f with $(X_1 - [n]_q) f$ which has polynomial coefficients in X_3 of lower degree. Inductively we may replace f with a nonzero element $\sum_{\substack{i_2 \geqslant 1 \\ \text{finite sum}}} X_2^{i_2} f_{i_2}(X_1) \in I \setminus \{0\}$. Using the same techniques on the right hand side to reduce the X_1–degree, we deduce that $I \bigcap X_2 \mathbb{C}[X_2] \neq \{0\}$. //

4.2.14 Corollary

Let R be a diffusion algebra of type III(a) or III(c), then R is a prime ring. Moreover, every nonzero prime ideal contains $X_2(X_2 - \eta)$ for some $\eta \in \mathbb{C} \setminus \{0\}$.

Proof:

The first part follows from the fact that the zero ideal is maximal with respect to being disjoint from the multiplicative subset $X_2 \mathbb{C}[X_2] \setminus \{0\}$. The second part comes from the fact that we know every ideal contains an element $X_2 f(X_2)$. If $f(X_2)$ factorises

$f(X_2) = f_1(X_2)f_2(X_2)$, observe that

$$\begin{aligned}(RX_2f_1(X_2)R)(RX_2f_2(X_2)R) &= Rf_1(X_2)(X_2RX_2)f_2(X_2)R \\ &\overset{(4.2.5)}{=} Rf_1(X_2)f_2(X_2)X_2^2\mathbb{C}[X_2]R \\ &\subseteq RX_2f(X_2)R,\end{aligned}$$

thus a prime ideal contains $X_2f(X_2)$ for an irreducible f (moreover, if $f(X_2) = X_2$ then, using the same trick, our prime ideal contains X_2 and so certainly contains $X_2(X_2 - \eta)$).

//

4.2.15 Proposition

The prime spectra of type III(a) and III(c) diffusion algebras are as pictured:

- *Type III(a):*

$$\begin{tikzcd}[arrows=dash]
R(X_1+\gamma)R + R(X_1+X_3+\kappa)R + RX_2R \arrow[d] & \textit{where } \gamma \in \mathbb{C} \\
R(X_1+X_3+\kappa)R + RX_2R \arrow[d] & \\
RX_2R \arrow[d] & \\
R(X_2(X_2-\eta))R \arrow[d] & \textit{where } \eta \in \mathbb{C}\setminus\{0\} \\
0 &
\end{tikzcd}$$

- *Type III(c):*

$$\begin{tikzcd}[arrows=dash]
R((q-1)X_3X_1+X_1+X_3+\kappa)R + RX_2R + R\left(X_3 - \frac{1}{1-q} + \eta\right)R \arrow[d] & \textit{where } \eta \in \mathbb{C}\setminus\{0\} \\
R((q-1)X_3X_1+X_1+X_3+\kappa)R + RX_2R \arrow[d] & \\
RX_2R \arrow[d] & \\
R(X_2(X_2-\eta))R \arrow[d] & \textit{where } \eta \in \mathbb{C}\setminus\{0\} \\
0 &
\end{tikzcd}$$

Proof:

After (4.2.5) and (4.2.14), it suffices to show that every ideal of $\frac{R}{RX_2(X_2-\eta)R}$ (for any $\eta \in \mathbb{C} \setminus \{0\}$) contains X_2. As $X_2\mathbb{C} \setminus \{0\}$ is a multiplicative set modulo $RX_2(X_2-\eta)R$ this would imply that the ideals $RX_2(X_2-\eta)R$ are prime for $\eta \in \mathbb{C} \setminus \{0\}$. Moreover, every ideal strictly containing $RX_2(X_2-\eta)R$ would contain RX_2R which is known to be prime and its prime factor ring is understood. Let $f \in I \setminus \{0\}$, a nonzero ideal of $\frac{R}{RX_2(X_2-\eta)R}$. Write $f = g + h$ where $g = \sum_{\substack{i_1, i_3 \in \mathbb{Z}^+ \\ \text{finite sum}}} g_{i_1 i_3} X_3^{i_3} X_1^{i_1}$ and $h = \sum_{\substack{j_1, j_3 \in \mathbb{Z}^+ \\ \text{finite sum}}} h_{j_1 j_3} X_3^{j_3} X_2 X_1^{j_1}$ (it is clear that sums of this type span this factor ring. Moreover, since $X_1X_2 = X_2X_3 = 0$ the new overlap ambiguities are resolveable and we in fact have a basis).

If $h = 0$, we use a trick from the previous proposition: we may replace g with a power of $((q-1)X_3X_1 + X_3 + X_1 + \kappa)$ and then multiply on either side by X_2 to obtain a nonzero scalar mutliple of $X_2^2 = \eta X_2$, therefore $X_2 \in I$.

If $h \neq 0$, then without loss of generality $g = 0$: if $g \neq 0$, we again we borrow the trick, from the preceding proposition of replacing g with $g' := ((q-1)X_3X_1 + X_3 + X_1 + \kappa)^m$ for some m and replace f with $f' = g' + h'$ if $h' = 0$ then we, again multiply on both sides by X_2 to obtain $X_2 \in I$, otherwise we replace f' with $f'' := [f', g']$ (for type III(a)) or $f'' := [f', X_1]_{q^m}$ (for type III(c)). In either case, the monomials of X_2–degree zero are killed and the monomials of maximal X_1–degree among monomials of X_2–degree one survive in X_1–degree one higher.

Write $f'' := \sum_{\substack{j_1 \in \mathbb{Z}^+ \\ \text{finite sum}}} f_{j_1}(X_3) X_2 X_1^{j_1}$, where $f_{j_1} \in \mathbb{C}[X_3]$. If any of the constant terms $f_{j_1 0}$ are nonzero, we premultiply by X_2; otherwise we deduce that none of the f_{j_1} are q–bad polynomials (4.2.11) and premultiply by $(X_1 - [n]_q)$. As in the previous proposition, by (4.2.9) we inductively deduce that $f''' := X_2 p(X_1) \in I \setminus \{0\}$ where $p(X_1) \in \mathbb{C}[X_1]$. Using the same trick on the right to reduce the X_1–degree, we deduce that $X_2 \in I \setminus \{0\}$, as required.

//

4.2.16 Proposition

Let R be a type III(b) or III(d) diffusion algebra, then every prime ideal contains either X_2 or $P_{n+1}(X_3)$ where $[n]_q + \kappa = 0$.

Proof:

Observe that P_{n+1} is the monic q–bad polynomial of minimal degree such that it has zero constant term. We first show that $X_2 R P_{n+1}(X_3)X_2 = 0$, it suffices to show that $X_2 X_3^{i_3} X_2^{i_2} X_1^{i_1} P_{n+1}(X_3)X_2 = 0$. We know that X_1 acts on $P_{n+1}(X_3)X_2$ by scalar multiplication by $[n+1]_q \neq 0$ (by the definition of a q–bad polynomial) so it suffices to examine $X_2 X_3^{i_3} X_2^{i_2} P_{n+1}(X_3)X_2 = X_2 X_3^{i_3} X_2^{i_2} X_3 P_n(X_3)X_2$ since, by (4.2.11), $P_{n+1} = \left(X_3 + \frac{[n]_q+\kappa}{q^n}\right) P_n(X_3)$ and $[n]_q+\kappa = 0$. Now, $X_2 X_3^{i_3} X_2^{i_2} X_3 P_n(X_3) X_2 = 0$ since $X_2 X_3 = 0$. This shows that every prime ideal contains either X_2 or $P_{n+1}(X_3)X_2$, we now show that $P_{n+1}(X_3)RX_2 \equiv 0 \mod RP_{n+1}(X_3)X_2R$. It suffices to consider $P_{n+1}(X_3)X_3^{i_3}X_2^{i_2}X_1^{i_1}X_2$ either $i_1 \neq 0$ in which case this is zero as $X_1X_2 = 0$ or $i_1 = 0$ in which case this is a multiple of $P_{n+1}(X_3)X_2$. Thus every prime contains either X_2 or $P_{n+1}(X_3)$. //

4.2.17 Lemma

Let R be a type III(b) or III(d) diffusion algebra (i.e. $\kappa = -[n]_q$, where $n \geqslant 1$), then

$$[X_1, \frac{P_{n+1}(X_3)}{P_i(X_3)} \frac{P_{n+1}(X_1)}{P_{n+1-i}(X_1)}]_{q^{n+1-i}} = q^{n-i}[n+1-i]_q \frac{P_{n+1}(X_3)}{P_{i+1}(X_3)} \frac{P_{n+1}(X_1)}{P_{n-i}(X_1)} + \text{terms of lower } X_1\text{–degree,}$$

for $i \in \{0, 1, \ldots, n\}$ with the convention that $P_0 = 1$.

Proof:

Since $\kappa = -[n]_q$, $\frac{P_{n+1}(X)}{P_i(X)} = X(X-[1]_q)\dots(X-[n-i]_q)$ also, from the defining relations we deduce $X_1(X_3-[j]_q) = q(X_3-[j-1])X_1 +$ terms of lower X_1–degree. Thus:

$$X_1\frac{P_{n+1}(X_3)}{P_i(X_3)}\frac{P_{n+1}(X_1)}{P_{n+1-i}(X_1)} = X_1X_3(X_3-[1]_q)\dots(X_3-[n-i]_q)X_1(X_1-[1]_q)\dots(X_1-[i-1]_q)$$

$$= \big(q^{n+1-i}(X_3+q^{-1})X_3(X_3-[1]_q)\dots(X_3-[n-i-1]_q)X_1+ \text{terms of lower } X_1\text{–degree}\big)\, X_1(X_1-[1]_q)\dots(X_1-[i-1]_q)$$

$$= \big(q^{n+1-i}(X_3-[n-i]_q+q^{-1}[n+1-i]_q)X_3(X_3-[1]_q)\dots(X_3-[n-i-1]_q)X_1+ \text{terms of lower } X_1\text{–degree}\big)\, X_1(X_1-[1]_q)\dots(X_1-[i-1]_q)$$

$$= q^{n+1-i}\frac{P_{n+1}(X_3)}{P_i(X_3)}\frac{P_{n+1}(X_1)}{P_{n+1-i}(X_1)}X_1+$$

$$q^{n-i}[n+1-i]_qX_3(X_3-[1]_q)\dots(X_3-[n-i-1]_q)X_1(X_1-[1]_q)\dots(X_1-[i-1]_q)X_1$$

$$+ \text{ terms of lower } X_1\text{–degree}$$

$$= q^{n+1-i}\frac{P_{n+1}(X_3)}{P_i(X_3)}\frac{P_{n+1}(X_1)}{P_{n+1-i}(X_1)}X_1 + q^{n-i}[n+1-i]_q\frac{P_{n+1}(X_3)}{P_{i+1}(X_3)}\frac{P_{n+1}(X_1)}{P_{n-i}(X_1)}$$

$$+ \text{ terms of lower } X_1\text{–degree}$$

//

4.2.18 Corollary

With R as above, every element of the subfactor $\frac{\mathbb{C}[X_3,X_1]}{P_{n+1}(X_3)}$ has a representative of total degree at most n.

Proof:

We prove the claim that every monomial of total degree at least $n+1$ is congruent modulo $P_{n+1}(X_3)$ to an element of total degree at most n and of at most the same X_1–degree. The bad polynomial itself, $P_{n+1}(X_3)$ gives us a rule for reducing any monomial of X_1–degree zero and total degree at least $n+1$ to an element of X_1–degree zero and total degree at most n. By the preceeding lemma, $\frac{[X_1,P_{n+1}(X_3)]_{q^{n+1}}}{q^n[n+1]_q} = \frac{P_{n+1}(X_3)P_{n+1}(X_1)}{P_1(X_3)P_n(X_1)} +$

terms of X_1–degree zero $=: P'_{n+1}(X_3, X_1)$ is in the ideal generated by $P_{n+1}(X_3)$ thus we get a rule for reducing any monomial of X_1–degree one and total degree at least $n+1$ to an element of X_1–degree at most one and total degree at most n. Again, by the preceding lemma, $\frac{[X_1, P'_{n+1}(X_3,X_1)]_{q^n}}{q^{n-1}[n]_q} = \frac{P_{n+1}(X_3)P_{n+1}(X_1)}{P_2(X_3)P_{n-1}(X_1)} +$ terms of lower X_1–degree $=: P''_{n+1}(X_3, X_1)$ is in the ideal generated by $P_{n+1}(X_3)$ and thus gives us a rule for reducing monomials of X_1–degree 2. Iteratively defining $P^{(j+1)}_{n+1}(X_3, X_1) := \frac{[X_1, P^{(j)}_{n+1}(X_3,X_1)]_{q^{n+2-j}}}{q^{n+1-j}[n+2-j]_q}$, we eventually deduce rules for reducing monomials of X_1–degree at most $n+1$. Observe also that $P^{(n+1)}_{n+1}(X_3, X_1) = P_{n+1}(X_1) +$ terms of lower X_1–degree so we get a rule for reducing the degree of any monomial of degree at least $n+1$.

//

4.2.19 Proposition

Let R be a type III(b) or III(d) diffusion algebra, let I be a nonzero ideal of $\frac{R}{RP_{n+1}(X_3)R}$ then $I \bigcap X_2\mathbb{C}[X_2] \neq \{0\}$.

Proof:

Let $\overline{f} \in I \setminus \{0\}$, write $\overline{f} = \overline{g} + \overline{h}$ where $\overline{g} \in \frac{\mathbb{C}[X_3,X_1]}{P_{n+1}(X_3)}$ and $\overline{h}$ be a sum of monomials all of which have degree at least one in X_2. Without loss of generality $\overline{g} = 0$, otherwise we may replace it with $\overline{((q-1)X_3X_1 + X_3 + X_1 + \kappa)}^n$ (this follows from (4.2.8) in case III(d) and from iteratively taking commutators with $X_3 + X_1 + \kappa$ in case III(b)) where $\kappa = -[n]_q$. We may then multiply on either side by X_2 (see (4.2.12)) to deduce that without loss of generality $\overline{g} = 0$. Write $\overline{f} = \sum_{\substack{i_2 \in \mathbb{N} \\ i_1 \in \{0,1,\ldots,n\} \\ \text{finite sum}}} f_{i_2 i_3}(\overline{X_3})\overline{X_2}^{i_2}\overline{X_1}^{i_1}$ where we may assume the polynomials $f_{i_2 i_3}(\overline{X_3})$ have degree at most n. If any of these polynomials are bad polynomials then they must have a nonzero constant term thus we may apply our algorithm to reduce the X_3–degree. Applying the same trick on the right to X_1 yields the result.

//

Note that we have emphasised that elements are images in the factor algebra using bar notation as we have yet to establish a basis for this factor, merely a spanning set. We shall suppress this notation in the following propositions.

4.2.20 Corollary

With R as above, the ideal $RP_{n+1}(X_3)R$ is prime. Moreover any prime containing $RP_{n+1}(X_3)R$ also contains $X_2(X_2-\eta)$ for some $\eta \in \mathbb{C} \setminus \{0\}$.

Proof:

The ideal $RP_{n+1}(X_3)R$ is maximal with respect to not intersecting the multiplicative set $X_2\mathbb{C}[X_2] \setminus \{0\}$, thus is prime. Moreover, $X_2(X_2-\eta)\frac{R}{RP_{n+1}(X_3)R}X_2f(X_2) = X_2\mathbb{C}[X_2]X_2(X_2-\eta)f(X_2)$ so if $X_2(X_2-\eta)f(X_2) \in P$ for any prime, P, containing $RP_{n+1}(X_3)R$ then either $X_2(X_2-\eta)$ also is or $X_2f(X_2)$ is and we may apply induction.

//

4.2.21 Lemma

With R as above the factor ring $\frac{R}{RP_{n+1}(X_3)R+RX_2(X_2-\eta)R}$ has dimension at most $\frac{(n+1)(3n+4)}{2}$.

Proof:

We have already seen that in X_2–degree zero we have a reduction rule for any monomial of total degree bigger than n thus this subspace of our algebra has dimension at most $\frac{(n+1)(n+2)}{2}$. In X_2–degree one we have monomials of the form $X_3^iX_2X_1^j$ where we may assume that $i, j \leqslant n$ thus the whole algebra has dimension at most $(n+1)^2 + \frac{(n+1)(n+2)}{2} = \frac{(n+1)(3n+4)}{2}$. //

4.2.22 Lemma

With R as above, the polynomial $(q-1)X_3+1$ is invertible modulo $P_{n+1}(X_3)$ for all $n \in \mathbb{N}$.

Proof:

We know that $\frac{1}{1-q}$ is not a root of the bad polynomial, $P_{n+1}(X_3)$ and can therefore use long division of polynomials with remainder. //

4.2.23 Proposition

Let R be a type III(b) or III(d) diffusion algebra, then the map

$$\rho_\eta : \frac{R}{RP_{n+1}(X_3)R + RX_2(X_2-\eta)R} \longrightarrow M_{n+1}(\mathbb{C})$$

given by:

$$X_3 \longrightarrow \mathrm{diag}([n]_q,[n-1]_q,\dots,[1]_q,0)$$

$$X_2 \longrightarrow \begin{array}{cl} \left(\begin{array}{ccc|c} 0 & \dots & 0 & (-1)^n\frac{[n]_{q_2}!}{0!}\eta \\ & & & \vdots \\ \vdots & & \vdots & (-1)^{n+1-j}\frac{[n]_{q_2}!}{[j-1]_{q_2}!}\eta \\ & & & \vdots \\ 0 & \dots & 0 & \eta \end{array}\right) & \begin{array}{l} \\ \\ \longleftarrow \textit{row } j \\ \\ \\ \end{array} \\ \begin{array}{cccc} & & & \qquad\quad | \\ & & & \qquad\quad \textit{col } n+1. \end{array} & \end{array}$$

$$X_1 \longrightarrow \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & [1]_q & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & [n-1]_q & 0 \\ 0 & \dots & 0 & 1 & [n]_q \end{pmatrix}$$

is a surjective representation.

Proof:

It is clear that $P_{n+1}(\rho_\eta(X_3)) = 0$ since P_{n+1} is the characteristic polynomial of $\rho_\eta(X_3)$. It is also easy to see that $\rho_\eta(X_2)\left(\rho_\eta(X_2) - \eta\mathrm{Id}_{n+1}\right) = 0$ and that $\rho_\eta(X_1)\rho_\eta(X_2) = \rho_\eta(X_2)\rho_\eta(X_3) = 0$. We now verify that the final defining relation is satisfied:

$$
\begin{aligned}
&\rho_\eta(X_1)\rho_\eta(X_3) - q\rho_\eta(X_3)\rho_\eta(X_1)\\
&= \left(\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & [1]_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & [n]_q \end{pmatrix} + \left(\begin{array}{c|c} 0\dots0 & 0 \\ \hline & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{array}\right)\right) \begin{pmatrix} [n]_q & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & [1]_q & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}\\
&\quad - q \begin{pmatrix} [n]_q & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & [1]_q & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & [1]_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & [n]_q \end{pmatrix} + \left(\begin{array}{c|c} 0\dots0 & 0 \\ \hline & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{array}\right)\right)\\
&= (1-q) \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & [1]_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & [n]_q \end{pmatrix} \begin{pmatrix} [n]_q & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & [1]_q & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}\\
&\quad + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ [n]_q & 0 & 0 & \dots & 0 \\ 0 & [n-1]_q & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & [1]_q & 0 \end{pmatrix}\\
&\quad - q \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ [n-1]_q & 0 & 0 & \dots & 0 \\ 0 & [n-2]_q & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}\\
&= (1-q)\mathrm{diag}(0, [1]_q[n-1]_q, [2]_q[n-2]_q, \dots, [n-1]_q[1]_q, 0)\\
&\quad + \left(\begin{array}{c|c} 0\dots0 & 0 \\ \hline & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{array}\right)
\end{aligned}
$$

whereas:

$$\rho_\eta(X_1) + \rho_\eta(X_3) - [n]_q \mathrm{Id}_n = (\mathrm{diag}(0, [1]_q, \ldots, [n]_q) + \mathrm{diag}([n]_q, \ldots, [1]_q, 0) - [n]_q \mathrm{Id}_{n+1}) + \left(\begin{array}{c|c} 0 \ldots 0 & 0 \\ \hline & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{array}\right).$$

Observe also that:

$$\begin{aligned} (1-q)[i]_q[n-i]_q &= (1-q)\left(\sum_{j=0}^{i-1} q^j\right)\left(\sum_{k=0}^{n-i-1} q^k\right) \\ &= \left(1-q^i\right)\left(\sum_{k=0}^{n-i-1} q^k\right) \\ &= \sum_{j=0}^{n-i-1} q^j - \sum_{k=i}^{n-1} q^k \\ &= \sum_{j=0}^{n-i-1} q^j - \sum_{k=0}^{n-1} q^k + \sum_{\ell=0}^{i-1} q^\ell \\ &= [i]_q + [n-i]_q - [n]_q. \end{aligned}$$

So the final defining relation is satisfied. Finally we prove surjectivity. By the preceding lemma $(q-1)X_3+1$ is invertible modulo $P_{n+1}(X_3)$, the image under ρ_η of $\frac{[n]_q - X_3}{(q-1)X_3+1}$ is $\mathrm{diag}(0, [1]_q, \ldots, [n]_q)$. Thus the image of ρ_η contains $\left(\begin{array}{c|c} 0 \ldots 0 & 0 \\ \hline & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{array}\right) = \rho_\eta(X_1 - \frac{[n]_q - X_3}{(q-1)X_3+1})$. It is clear that repeatedly pre– and post–multiplying $\rho_\eta(X_2)$ by this matrix allows us to generate the whole of $M_{n+1}(\mathbb{C})$. //

4.2.24 Lemma

The matrix, $\begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & q & q^2 & \dots \\ 1 & q^2 & q^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ *can be transformed by means of row and column operations (involving only rows above, or columns to the left of, the one being transformed) to the identity matrix.*

Proof:

First observe:

$$\begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & q & q^2 & \dots \\ 1 & q^2 & q^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \overrightarrow{(r_2'=r_2-r_1)\circ(r_3'=r_3-r_1)\circ\dots} \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & q-1 & q^2-1 & \dots \\ 0 & q^2-1 & q^4-1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\overrightarrow{(r_2''=\frac{r_2'}{q-1})\circ(r_3''=\frac{r_3'}{q^2-1})\circ\dots} \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & [1]_q & [2]_q & \dots \\ 0 & [1]_{q^2} & [2]_{q^2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\overrightarrow{(c_2'=c_2-c_1)\circ(c_3'=c_3-c_1)\circ\dots} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & [1]_q & [2]_q & [3]_q & \dots \\ 0 & [1]_{q^2} & [2]_{q^2} & [3]_{q^2} & \dots \\ 0 & [1]_{q^3} & [2]_{q^3} & [3]_{q^3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\overrightarrow{(c_3''=c_3'-c_2')\circ(c_4''=c_4'-c_3''-c_2')\circ(c_5''=c_5'-c_4''-c_3''-c_2')\circ\dots} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & q & q^2 & \dots \\ 0 & 1 & q^2 & q^4 & \dots \\ 0 & 1 & q^3 & q^6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\overrightarrow{(c_3'''=\frac{c_3''}{q})\circ(c_4'''=\frac{c_4''}{q^2})\circ\dots} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 0 & 1 & q & q^2 & \dots \\ 0 & 1 & q^2 & q^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We may now iterate this algorithm on the submatrix obtained by deleting the first row and column. Eventually we will reach the identity matrix. //

4.2.25 Proposition

Let R be a type III(b) or III(d) diffusion algebra, then the map

$$\rho'_\eta : \frac{R}{RP_{n+1}(X_3)R + RX_2(X_2-\eta)R} \longrightarrow M_{n+1}(\mathbb{C})$$

given by:

$$\begin{aligned}
X_3 &\longrightarrow \operatorname{diag}([n]_q, [n-1]_q, \ldots, [1]_q, 0) \\
X_2 &\longrightarrow 0 \\
X_1 &\longrightarrow \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & [1]_q & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & [n-1]_q & 0 \\ 0 & \ldots & 0 & 1 & [n]_q \end{pmatrix}
\end{aligned}$$

is a representation that surjects onto the subring of lower triangular matrices.

Proof:

It is clear from the previous proposition that this is indeed a representation. Also, from the previous proposition, we may deduce that the matrix, $\left(\begin{array}{c|c} 0\ldots 0 & 0 \\ \hline & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{array}\right)$, is in the image 0f ρ'_η. It now suffices to show that the subalgebra generated by $\rho'_\eta(X_3)$ has dimension $n+1$. For a type III(d) diffusion algebra we observe that $\rho'_\eta((q-1)X_3+1) = \operatorname{diag}(q^n, q^{n-1}, \ldots, 1)$, by the preceding lemma, the set $\left\{\operatorname{diag}(q^n i, q^{n-1} i, \ldots, 1)\right\}_{i\in\{0,1,\ldots,n\}}$ is linearly independent. For a type III(b) diffusion algebra we observe that the powers of the matrix, $\operatorname{diag}(n, n-1, \ldots, 1, 0)$, clearly generate a vector space of dimension n. In fact the powers of any real matrix, $\operatorname{diag}(x_{n-1}, \ldots, x_1, 1, 0)$, where $1 < x_1 < \cdots < x_{n-1}$, generate a vector space of dimension n as $\operatorname{diag}(x_{n-1}, \ldots, x_1, 1, 0)^2 - \operatorname{diag}(x_{n-1}, \ldots, x_1, 1, 0) = (x_{n-1}(x_{n-1}-1), \ldots, x_1(x_1-1), 0, 0)$ and $x_1(x_1-1) < x_2(x_2-1) < \cdots < x_{n-1}(x_{n-1}-1)$ and we may then rescale such that $x_1(x_1-1) \to 1$ and apply induction. Once we include the identity matrix we clearly have a vector space of dimension $n+1$. //

4.2.26 Lemma

In addition to the kernel (from now on denoted Q_η) of the representation ρ_η above there are also $n+1$ codimension 1 ideals of $R' := \frac{R}{RP_{n+1}(X_3)R+RX_2(X_2-\eta)R}$ of the form $R'X_2R' + R'(X_3-[j]_q)R' + R'(X_1-[n-j]_q)R'$ for $0 \leqslant j \leqslant n$.

Proof:

It suffices to show that the map

$$\begin{aligned} X_3 &\longrightarrow [j]_q \\ X_2 &\longrightarrow 0 \\ X_1 &\longrightarrow [n-j]_q \end{aligned}$$

is a homomorphism (as then it would clearly surject onto the complex numbers, a prime ring). It clearly respects the relations $X_2^2 = \eta X_2$, $X_1X_2 = 0$ and $X_2X_3 = 0$. $[j]_q$ is a root of $P_{n+1}(X_3)$ for all $0 \leqslant j \leqslant n$ so it satisfies the relation $P_{n+1}(X_3) = 0$. Finally, the relation $X_1X_3 - qX_3X_1 = X_3 + X_1 - [n]_q$ becomes $(1-q)[j]_q[n-j]_q = ([j]_q + [n-j]_q - [n]_q)$, an identity we verified in proposition (4.2.23). The kernels of these homomorphisms will be denoted $P_{j\eta}$ the homomorphisms themselves will be denoted $\chi_{j\eta}$. //

4.2.27 Proposition

The prime spectra of type III(b) and III(d) diffusion algebras are as pictured:

- *Type III(b):*

$$\begin{tikzcd}[column sep=small] R(X_1+\gamma)R + R(X_1+X_3-n)R + RX_2R \arrow[d, dash] & Q_\eta \arrow[dd, dash] & \textit{where } \gamma \in \mathbb{C}, \eta \in \mathbb{C}\setminus\{0\} \\ R(X_1+X_3-n)R + RX_2R \arrow[d, dash] & & \\ RX_2R & RP_{n+1}(X_3)R & \end{tikzcd}$$

- *Type III(d):*

$$\begin{tikzcd}
R\left((q-1)X_3X_1+X_1+X_3-[n]_q\right)R+RX_2R+R\left(X_3-\frac{1}{1-q}+\eta\right)R \arrow[d, dash] & Q_\eta \arrow[dd, dash] & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
R\left((q-1)X_3X_1+X_1+X_3-[n]_q\right)R+RX_2R \arrow[d, dash] & & \\
RX_2R & RP_{n+1}(X_3) &
\end{tikzcd}$$

Proof:

With the notation of the preceding lemma, it suffices to show that the ideals $P_0, P_1, \dots, P_n, Q$ (dropping, for convenience, the subscript η) are all the prime ideals of R'. Consider the representation $\rho_\eta \oplus \rho'_\eta$. This representation surjects onto the subring of $M_{2(n+1)}(\mathbb{C})$ consisting of matrices of the shape

$$\left(\begin{array}{ccc|cccc}
* & \dots & * & & & & \\
\vdots & & \vdots & & 0 & & \\
* & \dots & * & & & & \\
\hline
 & & & * & 0 & \dots & 0 \\
 & 0 & & * & * & \ddots & \vdots \\
 & & & \vdots & \ddots & \ddots & 0 \\
 & & & * & \dots & * & *
\end{array}\right)$$

where the horizontal and vertical subdivisions distinguish between the ρ_η summand and he ρ'_η summand. In view of our upper bound for the dimension, (4.2.21), we see that we must have equality and $\rho_\eta \oplus \rho'_\eta$ must be an isomorphism. The ideal in R' corresponding to matrices of the form

$$\left(\begin{array}{c|cccc}
0 & & 0 & & \\
\hline
 & 0 & 0 & \dots & 0 \\
0 & * & 0 & \dots & 0 \\
 & \vdots & \ddots & \ddots & \vdots \\
 & * & \dots & * & 0
\end{array}\right)$$

is a nilpotent ideal, N, with $\frac{R'}{N} \cong \frac{R'}{Q} \oplus \frac{R'}{P_n} \oplus \dots \oplus \frac{R'}{P_0}$. N is nilpotent so it is contained in any prime, in particular it is contained in $Q \cap P_n \cap \dots \cap P_0$. Since R' is Artinian, $Q, P_n, \dots, P_0$

are all maximal, thus

$$\frac{R'}{Q \cap P_n \cap \cdots \cap P_0} \cong \frac{R'}{Q} \oplus \frac{R'}{P_n} \oplus \cdots \oplus \frac{R'}{P_0}$$

and hence $N = Q \cap P_n \cap \cdots \cap P_0$. Every prime contains N so it contains $Q \cap P_n \cap \cdots \cap P_0$ so it contains $QP_n \ldots P_0$ so it contains one of $Q, P_n, \ldots, P_0$ and, since a prime Artinian ring is simple, it must equal one of $Q, P_n, \ldots, P_0$. //

We move on now to examine the prime spectrum of R when R is a type IV diffusion algebra.

4.2.28 Proposition

The Prime spectrum of a type IV(a) diffusion algebra, R, is as pictured:

$$\begin{tikzcd}
\begin{matrix} R(X_1+\gamma)R+ \\ RX_2R+RX_3R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} RX_1R+RX_3R \\ +R(X_2+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} RX_1R+R(X_2+1)R \\ +R(X_3+\gamma)R \end{matrix} \arrow[d, dash] & \textit{where } \gamma \in \mathbb{C} \\
RX_2R+RX_3R & & RX_1R+RX_3R & & R(X_2+1)R+RX_1R &
\end{tikzcd}$$

Proof:

First observe that $(RX_1R)(RX_3R) = (RX_1)(X_3R) = RX_1X_3R = 0$. Moreover, $\frac{R}{RX_1} \cong \mathbb{C}\langle (X_2+1), X_3 : (X_2+1)X_3 = 0\rangle$ and $\frac{R}{X_3R} \cong \mathbb{C}\langle X_1, X_2 : X_1X_2 = 0\rangle$, the result follows. //

4.2.29 Proposition

The Prime spectrum of a type IV(b) diffusion algebra, R, is as pictured:

$$\begin{tikzcd}
\begin{matrix} RX_1R+RX_3R \\ +R(X_2+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} RX_1R+RX_2R \\ +R(X_3+\gamma)R \end{matrix} \arrow[d, dash] & \cdots & \begin{matrix} R(X_1+\gamma)R+ \\ RX_2R+R(X_3+1)R \end{matrix} \arrow[d, dash] & \textit{where } \gamma \in \mathbb{C} \\
RX_1R+RX_3R & & RX_1R+RX_2R & & RX_2R+R(X_3+1)R &
\end{tikzcd}$$

Proof:

Observe that $RX_1R + RX_3R = RX_1 + X_3R$. Now $RX_2R(RX_3R + RX_1R)RX_2R = RX_2RX_1X_2R + RX_2X_3RX_2R = 0$ thus any prime contains either X_2 or both X_1 and X_3.

Moreover, $\frac{R}{RX_2R} \cong \mathbb{C}\langle X_1, (X_3+1) : X_1(X_3+1) = 0\rangle$, the result follows. //

4.2.30 Remark

Diffusion algebras of types IV(c), IV(d) and IV(e) can be summarised by the presentation:

$$X_1X_2 = \kappa_1X_2 \tag{4.2.30.1a}$$

$$X_1X_3 = 1 \tag{4.2.30.1b}$$

$$X_2X_3 = \kappa_3X_2 \tag{4.2.30.1c}$$

$$\tag{4.2.30.1}$$

Where $\kappa_1\kappa_3 \neq 1$. We will use this presentation to analyse their prime spectra.

4.2.31 Lemma

Let R be given by the presentation (4.2.30.1)*, then every nonzero ideal of R intersects the set $X_2\mathbb{C}[X_2]$.*

Proof:

Let I be a nonzero ideal of R and $f \in I \setminus \{0\}$. Write $f = g + h$ where g is in the subalgebra generated by X_1 and X_3 and h is a $\mathbb{C}$–linear combination of monomials of degree at least one in X_2. We claim that without loss of generality $g = 0$. If $g \neq 0$ we will consider $fX_3^kX_2$. If this is nonzero, it must be an element of I of the required form (i.e. all its monomials are of X_2–degree at least one). Moreover, since the relations are homogeneous in X_2, it will suffice to guarantee that $gX_3^kX_2 \neq 0$, we will describe an algorithm for choosing such a $k \in \mathbb{Z}^+$.

Grade $\mathbb{C}[X_1, X_3]$, the subalgebra generated by X_1 and X_3, by $\deg X_1 := -1, \deg X_3 := 1$. Let k be the smallest positive integer such that all homogeneous components of gX_3^k consist of at most two monomials none of which containing more than one X_1 as a factor and that at least one power of X_3 occurs somewhere with nonzero coefficient. If $\kappa_1 = 0$ then $gX_3^kX_2 \neq 0$. If $\kappa_1 \neq 0$ there are two possibilities. If there is a unique monomial of maximal X_3–degree, then $gX_3^kX_2 \neq 0$; otherwise there are two monomials of maximal X_3–degree, X_3^m and $X_3^mX_1$, say. In which case gX_3^{k+1} is nonzero and has a unique monomial of maximal X_3–degree, thus $gX_3^{k+1}X_2 \neq 0$. Thus, without loss of generality, our ideal I contains a nonzero element which is a $\mathbb{C}$–linear combination of monomials all of which have degree at least one in X_2.

Let $f = \sum_{jk} p_{jk}(X_3)X_2^jX_1^k$ $(j \geqslant 1)$ and let $n := \max_{jk}\{\deg(P_{jk}(X_3))\}$ then $X_1^{n-1}f$ is a nonzero sum of the same kind but where all the polynomials in X_3 have degree at most one. If $\kappa_1 = 0$ then we premultiply by X_1 to obtain a nonzero sum where all the p_{jk}s are scalars. If $\kappa_1 \neq 0$ then we observe that (upto scalar multiplication) $\kappa_1X_3 - 1$ is the unique linear polynomial, p, such that $X_1p(X_3)X_2 = 0$ if all the linear coefficients of $X_1^{n-1}f$ are scalar multiples of this polynomial then we premultiply by X_2 (we have $X_2(\kappa_1X_3 - 1)X_3 = (\kappa_1\kappa_2 - 1)X_2^2 \neq 0$ since $\kappa_1\kappa_2 \neq 1$) otherwise we premultiply by X_1 in either case we get a nonzero sum where all the p_{jk}s are scalars.

By symmetry, we may apply the same tricks on the other side to reduce the degree in X_1 to zero yielding a nonzero element of $X_2\mathbb{C}[X_2]$. //

4.2.32 Corollary

We have that R is a prime ring and every prime ideal contains $X_2(X_2 - \eta)$ for some $\eta \in \mathbb{C} \setminus \{0\}$.

Proof:

The zero ideal is maximal among ideals disjoint from the multiplicative set $X_2\mathbb{C}[X_2] \setminus \{0\}$, hence is prime. Moreover, as $X_2RX_2 = X_2^2\mathbb{C}[X_2]$ we may conclude that any prime contains $X_2(X_2 - \eta)$ for some η (this is because $(RX_2f(X_2)R)(RX_2g(X_2)R) \subseteq RX_2f(X_2)g(X_2)R$).

//

4.2.33 Lemma

With $R' := \frac{R}{RX_2(X_2-\eta)R}$ where $\eta \in \mathbb{C} \setminus \{0\}$, R' is not a prime ring and every prime ideal contains either X_2 or $f_\eta := (\kappa_1X_3 - 1)X_2(\kappa_3X_1 - 1) + \eta(\kappa_1\kappa_3 - 1)(1 - X_3X_1)$.

Proof:

Observe that $X_1 f_\eta = 0$, thus $X_2 R' f_\eta = X_2 \frac{\mathbb{C}[X_2]}{(X_2(X_2-\eta))} f_\eta = 0$ since $X_2 f_\eta = 0$. //

4.2.34 Proposition

The Prime spectrum of R, a diffusion algebra of one of the types IV(c), IV(d) and IV(e) is as pictured:

- *Type IV(c):*

$$\begin{tikzcd}
 & R\left(X_1+\eta\right)R + RX_2R + R\left(X_3X_1-1\right)R \arrow[d, dash] & & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
 & RX_2R + R\left(X_3X_1-1\right)R \arrow[dl, dash] \arrow[dr, dash] & & \\
RX_2R \arrow[dr, dash] & & \begin{array}{c} RX_2\left(X_2-\eta\right)R+ \\ R\left(X_2(X_1-1)+\eta(1-X_3X_1)\right)R \end{array} \arrow[dl, dash] & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
 & 0 & &
\end{tikzcd}$$

- *Type IV(d):*

$$\begin{tikzcd}
 & R\left(X_1+\eta\right)R + RX_2R + R\left(X_3X_1-1\right)R \arrow[d, dash] & & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
 & RX_2R + R\left(X_3X_1-1\right)R \arrow[dl, dash] \arrow[dr, dash] & & \\
RX_2R \arrow[dr, dash] & & \begin{array}{c} RX_2\left(X_2-\eta\right)R+ \\ R\left(X_2-\eta(1-X_3X_1)\right)R \end{array} \arrow[dl, dash] & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
 & 0 & &
\end{tikzcd}$$

- *Type IV(e):*

$$\begin{tikzcd}
 & R\left(X_1+\eta\right)R + RX_2R + R\left(X_3X_1-\kappa\right)R \arrow[d, dash] & & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
 & RX_2R + R\left(X_3X_1-\kappa\right)R \arrow[dl, dash] \arrow[dr, dash] & & \\
RX_2R \arrow[dr, dash] & & \begin{array}{c} RX_2\left(X_2-\eta\right)R+ \\ R\left((\kappa X_3-1)X_2(\kappa X_1-1)+\eta\left(\frac{1}{\kappa}-1\right)\left(1-\frac{X_3X_1}{\kappa}\right)\right)R \end{array} \arrow[dl, dash] & \text{where } \eta \in \mathbb{C}\setminus\{0\} \\
 & 0 & &
\end{tikzcd}$$

Proof:

After the previous lemma, it will suffice for us to examine the factor rings $\frac{R}{RX_2R} \cong \mathbb{C}\langle X_1, X_3 : X_1X_3 = 1\rangle$, which is known (as mentioned at the very start of this section), and, adopting from now on the notation of the preceding lemmas (including the presentation (4.2.30.1)), $\frac{R}{RX_2(X_2-\eta)R+Rf_\eta R}$. For a type IV(d) algebra, X_2 becomes superfluous as a generator in this case and the factor ring in question is again $\mathbb{C}\langle X_1, X_3 : X_1X_3 = 1\rangle$. We now address the remaining two cases.

If R is a type IV(e) algebra, the corresponding presentation as in (4.2.30.1) has $\kappa_1 \neq 0 \neq \kappa_3$ in this case $R' := \frac{R}{RX_2(X_2-\eta)R+Rf_\eta R}$ is spanned over $\mathbb{C}$ by the monomials $1, X_1, X_2, X_3, X_3^nX_2, X_2X_1^m, X_3^nX_1^m$ where $n, m \in \mathbb{N}$. We claim that any nonzero ideal contains X_2 (thus R' is prime and $\frac{R'}{R'YR'} \cong \mathbb{C}[X^{\pm 1}]$).

Let I be any nonzero ideal of R' and let f be a nonzero element of I. Write $f = a + bX_2 + X_3c(X_3)X_2 + X_2d(X_1)X_1 + e(X_3, X_1)$ there are now four cases

- if $c \neq 0$ then postmultiplying by a sufficiently large power of X_3 we may assume $f = c'(X_3)X_2 + e'(X_3, X_1)$ where we may also assume $c'(X_3)$ has nonzero constant term (if need be we may premultiply by a power of X_1 to guarantee $b \neq 0$ before postmultiplying by the power of X_3) and that $e'(X_3, X_1)$ has no constant term. Pre– and postmultiplying by $(1 - X_3X_1)$ gives us that $(1 - X_3X_1)X_2(1 - X_3X_1) \in I$.
- If $c = 0$ but $d \neq 0$ then we can do the same trick on the other side with X_1s instead of X_3s and again we may conclude that I contains $(1 - X_3X_1)X_2(1 - X_3X_1)$.
- If $c = d = 0$ but $b \in \mathbb{C} \setminus \{0\}$ then postmultiply by a sufficiently large power of X_3 to ensure that $a + e(X_3, X_1)$ no longer has a constant term then pre– and postmultiply by $(1 - X_3X_1)$ to, once again, deduce that I contains $(1 - X_3X_1)X_2(1 - X_3X_1)$.
- Finally if $b = c = d = 0$ then our ideal intersects the subalgebra generated by X and Z in which every ideal contains $(1 - ZX)$.

In any case our ideal contains $(1 - X_3X_1)X_2(1 - X_3X_1)$ and hence contains $X_2(1 - X_3X_1)X_2(1 - X_3X_1)X_2 = (1 - \kappa_1\kappa_2)^2\eta^2X_2$.

If R is a type IV(c) algebra, the corresponding presentation as in (4.2.30.1) has $\kappa_1 = 0, \kappa_3 = 1$ in this case $R' := \frac{R}{RX_2(X_2-\eta)R+Rf_\eta R}$ is spanned over $\mathbb{C}$ by monomials of the form $1, X_1, X_2, X_3, X_3^n X_2, X_3^n X_1^m$ where $n, m \in \mathbb{N}$ we again claim that every nonzero ideal contains X_2 (hence R' is prime and $\frac{R'}{R'X_2R'} \cong \mathbb{C}[X_1^{\pm 1}]$).

Let I be any nonzero ideal of R' be a nonzero ideal and let $f = p(X_3)X_2 + e(X_3, X_1)$ be a nonzero element of I. If $p \neq 0$ then without loss of generality it has nonzero constant term, postmultiplying by a sufficiently large power of X_3 then gives us a nonzero element of the form $p(X_3)X_2 + X_3E(X_3)$ premultiplying by $(1 - X_3X_1)$ then gives us $X_2 \in I$. If $p = 0$ then our ideal intersects the subalgebra generated by X_1 and X_3 hence contains $(1 - X_3X_1)$ and postmultiplying by X_2 we then obtain, once again, $X_2 \in I$. //

4.2.35 Proposition

The Prime spectrum of a type IV(f) diffusion algebra, R, is as pictured:

$$\begin{tikzcd}[arrows=dash]
RX_1R + R(X_2+\gamma)R + RX_3R \quad \cdots \arrow[dd] & \begin{array}{c} R(X_1 - 1 + \eta)R + RX_2R \\ +R(X_3X_1 - X_3 - X_1)R \end{array} \arrow[d] & \textit{where } \gamma \in \mathbb{C}, \eta \in \mathbb{C} \setminus \{0\} \\
 & \begin{array}{c} RX_2R \\ R(X_3X_1 - X_3 - X_1)R \end{array} \arrow[d] & \\
RX_1R + RX_3R & RX_2R &
\end{tikzcd}$$

Proof:

Observe that $RX_1R + RX_3R = RX_1 + X_3R$. Now $RX_2R(RX_3R + RX_1R)RX_2R = RX_2(RX_1 + X_3R)X_2R = RX_2RX_1X_2R + RX_2X_3RX_2R = 0$ thus any prime contains either X_2 or both X_1 and X_3. Moreover, $\frac{R}{RX_2R} \cong \mathbb{C}\langle (X_1 - 1), (X_3 - 1) : (X_1 - 1)(X_3 - 1) = 1 \rangle$, the result follows. //

4.2.36 Lemma

Let R be a type IV(g) diffusion algebra, then R is not a prime ring and every prime ideal contains either $(1 - (X_2 - 1)X_1)$ *or* $(1 - X_3(X_2 + \lambda))$.

Proof:

Observe that we have

$$\begin{aligned} (1-X_3(X_2+\lambda))X_3 &= 0 && \text{and} & X_1(1-(X_2-1)X_1) &= 0 \\ (X_2+\lambda)(1-X_3(X_2+\lambda)) &= 0 && & (1-(X_2-1)X_1)(X_2-1) &= 0 \\ (X_1+\frac{1}{1+\lambda})(1-X_3(X_2+\lambda)) &= 0 && & (1-(X_2-1)X_1)(X_3-\frac{1}{1+\lambda}) &= 0. \end{aligned}$$

Thus $(R(1-(X_2-1)X_1)R)(R(1-X_3(X_2+\lambda))R) = R(1-(X_2-1)X_1)(1-X_3(X_2+\lambda))R$ and

$$\begin{aligned} &(1-(X_2-1)X_1)(1-X_3(X_2+\lambda)) \\ &\qquad = (1-(X_2-1)X_1) - (1-(X_2-1)X_1)X_3(X_2+\lambda) \\ &\qquad = (1-(X_2-1)X_1) - \frac{1}{1+\lambda}(1-(X_2-1)X_1)(X_2+\lambda) \\ &\qquad = (1-(X_2-1)X_1) - \frac{1+\lambda}{1+\lambda}(1-(X_2-1)X_1) = 0 \end{aligned}$$

Thus R is not prime and every prime contains either $(1-(X_2-1)X_1)$ or $(1-X_3(X_2+\lambda))$.

//

We will now consider $R' := \frac{R}{R(1-(X_2-1)X_1)R} \cong \mathbb{C} < X_1^{\pm 1}, X_3 : X_1X_3 = \frac{1}{1+\lambda}(X_1 - X_3) >$. Observe that $\frac{R}{R(1-X_3(X_2+\lambda))R} \cong R'^{\mathrm{op}}$.

4.2.37 Lemma

Every nonzero ideal of R' contains $(1-X_3(X_1^{-1}+\lambda+1))$.

Proof:

Let $0 \neq I \lhd R'$ and let $f \in I \setminus \{0\}$. Write $f = \sum_{i=0}^{n} X_3^i p_i(X_1)$ with $p_i \in \mathbb{C}[X_1^{\pm 1}]$, $p_n(X_1) \neq 0$ then we may premultiply by $(X_1^{-1}+\lambda+1)^{n-1}$ to get an element of the form $f' = X_3 p_1'(X_1) + p_0'(X_1) \neq 0$ (since $(X_1^{-1}+\lambda+1)X_3 = 1$). At this point there are three cases:

- If $p_1' \neq 0$ and $(X_1^{-1} + \lambda + 1)f' \neq 0$ then $(1 - X_3(X_1^{-1} + \lambda + 1))(X_1^{-1} + \lambda + 1)f'$ is a nonzero element of I of the form $(1 - X_3(X_1^{-1} + \lambda + 1))p(X_1)$.

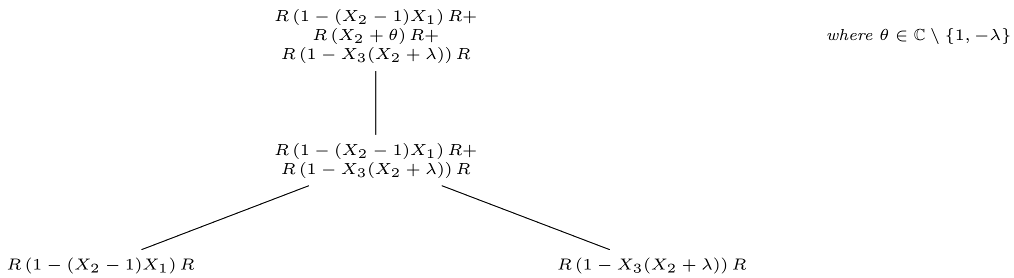
- If $p_1' = 0$ then we must have $p_0' \neq 0$ and then $(1 - X_3(X_1^{-1} + \lambda + 1))f'$ is a nonzero element of I of the form $(1 - X_3(X_1^{-1} + \lambda + 1))p(X_1)$.

- If $p_1' \neq 0$ and $(X_1^{-1} + \lambda + 1)f' = 0$ then we can conclude that $p_1'(X_1) = -(X_1^{-1} + \lambda + 1)p_0'(X_1)$. Thus f' is a nonzero element of I of the form $(1 - X_3(X_1^{-1} + \lambda + 1))p(X_1)$.

Without loss of generality $p(X_1)$ is a polynomial in the variable X_1^{-1} (as we may postmultiply by X_1^{-1} until no strictly positive powers of X_1 occur) we may then rewrite it as a polynomial q in the variable $(X_1^{-1} + \lambda + 1)$. Without loss of generality q has nonzero constant term, q_0, (as if $(X_1^{-1} + \lambda + 1)^i$ is the smallest power of $(X_1^{-1} + \lambda + 1)$ occurring we may postmultiply by X_3^i). Thus our ideal I contains $(1 - X_3(X_1^{-1} + \lambda + 1))q((X_1^{-1} + \lambda + 1))(1 - X_3(X_1^{-1} + \lambda + 1)) = q_0(1 - X_3(X_1^{-1} + \lambda + 1))^2 = q_0(1 - X_3(X_1^{-1} + \lambda + 1))$. Thus every nonzero ideal of R' contains $(1 - X_3(X_1^{-1} + \lambda + 1))$. //

4.2.38 Proposition

The Prime spectrum of a type IV(g) diffusion algebra, R, is as pictured:



Proof:

After the preceding lemma, R' is a prime ring since $(1 - X_3(X_1^{-1} + \lambda + 1))$ is an idempotent of R'. Therefore the singleton containing just this element is a multiplicative set and, by the previous proposition, $\{0\}$ is maximal among ideals disjoint from it thus is prime.

By symmetry in $R'' := \frac{R}{R(1-X_3(X_2+\lambda))R}$ every nonzero ideal contains $(1-(X_3^{-1}-\lambda-1)X_1)$. The ideals $R''(1-(X_3^{-1}-\lambda-1)X_1)R''$ and $R'(1-X_3(X_1^{-1}+\lambda+1))R'$ both correspond to the ideal $R(1-X_3(X_2+\lambda))R+R(1-(X_2-1)X_1)R$ of R. The factor ring by this ideal is

$$\begin{aligned}\frac{R'}{R'(X_3(X_1^{-1}+\lambda+1)-1)R'} &\cong \mathbb{C} < X_1^{\pm 1}, X_3 : (X_1^{-1}+\lambda+1)X_3 = X_3(X_1^{-1}+\lambda+1) = 1 > \\ &\cong \left(\mathbb{C}[X_1]_{\{X_1^j\}_{j\in\mathbb{Z}^+}}\right)_{\{(X_1^{-1}+\lambda+1)^i\}_{i\in\mathbb{Z}^+}} \\ &\cong \mathbb{C}[X_1]_{\{(X_1((\lambda+1)X_1+1))^i\}_{i\in\mathbb{Z}^+}}\end{aligned}$$

which is a prime ring whose nonzero primes are generated by $(X_1-\theta')$ for $\theta' \in \mathbb{C}\backslash\{0, -\frac{1}{1+\lambda}\}$.

//

4.2.39 Proposition

The Prime spectrum of a type IV(h) diffusion algebra, R, is as pictured:

$$\begin{tikzcd}[column sep=small]
RX_1R+RX_2R+R\,(X_3+\gamma)\,R \quad \cdots \arrow[dd, dash] & \begin{array}{c} R\,(X_1+\eta)\,R+RX_3R \\ +R\,(X_2X_1-X_2-X_1)\,R \end{array} \arrow[d, dash] & \text{where } \gamma\in\mathbb{C}, \eta\in\mathbb{C}\setminus\{0\} \\
 & \begin{array}{c} RX_3R+ \\ R\,(X_2X_1-X_2-X_1)\,R \end{array} \arrow[d, dash] & \\
RX_1R+RX_2R & RX_3R &
\end{tikzcd}$$

Proof:

Observe that $RX_3R = X_3R$, therefore $(RX_1R+RX_2R)RX_3R = RX_1X_3R + RX_2X_3R = 0$ thus any prime contains either X_3 or X_1 and X_2. Moreover, $\frac{R}{RX_3R} \cong \mathbb{C}\left\langle (X_1-1),(X_2-1) : (X_1-1)(X_2-1)=1\right\rangle$, this completes the result. //

We now look at the prime spectra of the type V diffusion algebras.

4.2.40 Proposition

The prime spectra of type V diffusion algebras are as pictured:

- *Type V(a):*

$$
\begin{tikzcd}
R(X_1+\gamma)R + RX_2R + RX_3R \arrow[dd, dash] & \cdots & RX_1R + RX_2R + R(X_3+\gamma)R \arrow[d, dash] & \cdots & RX_1R + R(X_2+\gamma)R + R\left(X_3 - \frac{1}{1-q}\right)R \arrow[d, dash] & \textit{where } \gamma \in \mathbb{C} \\
 & & RX_1R + RX_2R \arrow[dr, dash] & & RX_1R + R\left(X_3 - \frac{1}{1-q}\right)R \arrow[dl, dash] & \\
RX_2R + RX_3R & & & RX_1R & &
\end{tikzcd}
$$

- *Type V(b):*

$$
\begin{tikzcd}
RX_1R + R\left(X_3 - \frac{1}{1-q} + \eta\right)R + R((q-1)X_3X_2 - X_3 - X_2)R \arrow[d, dash] & \cdots & R(X_1+\gamma)R + RX_2R + RX_3R \arrow[dd, dash] & \textit{where } \gamma \in \mathbb{C}, \eta \in \mathbb{C} \setminus \{0\} \\
RX_1R + R((q-1)X_3X_2 + X_3 + X_2)R \arrow[d, dash] & & & \\
RX_1R & & RX_2R + RX_3R &
\end{tikzcd}
$$

Proof:

Observe that $RX_1R = RX_1$, therefore $RX_1R(RX_2R + RX_3R) = RX_1X_2R + RX_1X_3R = 0$ so any prime contains either X_1 or both X_2 and X_3. Moreover, for a type V(a) diffusion algebra,

$$\frac{R}{RX_1R} \cong \mathbb{C}\left\langle X_2, (X_3 - \frac{1}{1-q}) : X_2(X_3 - \frac{1}{1-q}) - q(X_3 - \frac{1}{1-q})X_2 = 0 \right\rangle$$

whereas for a type V(b) diffusion algebra

$$\frac{R}{RX_1R} \cong \mathbb{C}\left\langle ((1-q)X_2 - 1), (X_3 - \frac{1}{1-q}) : \left[(1-q)X_2 - 1), (X_3 - \frac{1}{1-q})\right]_q = 1 \right\rangle,$$

the result follows. //

We shall now classify the prime spectrum of the type VI diffusion algebra.

4.2.41 Proposition

The prime spectrum of the type VI diffusion algebra is as pictured:

$$\begin{tikzcd}[column sep=small]
\begin{array}{c} RX_1R + RX_3R \\ +R\,(X_2+\gamma)\,R \end{array} \arrow[d, dash] & \cdots & \begin{array}{c} RX_1R + RX_2R \\ +R\,(X_3+\gamma)\,R \end{array} \arrow[d, dash] & \cdots & \begin{array}{c} R\,(X_1+\gamma)\,R + RX_2R \\ +R\left(X_3 - \frac{1}{1-q}\right)R \end{array} \arrow[d, dash] & \textit{where } \gamma \in \mathbb{C} \\
RX_1R + RX_3R \arrow[d, dash] & & RX_1R + RX_2R \arrow[dll, dash] \arrow[drr, dash] & & RX_2R + R\left(X_3 - \frac{1}{1-q}\right)R \arrow[d, dash] & \\
RX_1R & & & & RX_2R &
\end{tikzcd}$$

Proof:

Observe that $RX_2R = X_2R$ (to see this, we may exchange X_2 and X_3 in the PBW–basis order, for example), thus $(RX_1R)(RX_2R) = RX_1X_2R = 0$. Moreover,

$$\frac{R}{RX_1R} \cong \mathbb{C}\langle X_2, X_3 : X_2X_3 - q_2X_3X_2 = 0\rangle$$

and

$$\frac{R}{RX_2R} \cong \mathbb{C}\left\langle X_1, (X_3 - \frac{1}{1-q}) : [X_1, (X_3 - \frac{1}{1-q})]_{q_1} = 0\right\rangle,$$

this completes the proof. //

We finally examine the prime spectrum of the type VII diffusion algebra.

4.2.42 Lemma

Let R be any ring, let $\mathbb{X} := \{x^n\}_{n\in\mathbb{Z}^+}$ where x is any normal non–zerodivisor then there is an injection

$$\{Q \in \operatorname{spec} R : Q \cap \mathbb{X} = \emptyset\} \hookrightarrow \operatorname{spec} R\mathbb{X}^{-1}$$

given by the usual idea of expansion and contraction.

A similar result is well-known for a Noetherian ring, this result allows us to use those methods in certain non–Noetherian settings.

Proof:

Let $P \in \{Q \in \operatorname{spec} R : Q \cap \mathbb{X} = \emptyset\}$ then $P\mathbb{X}^{-1}$ is known to be a right ideal. Consider $rx^{-n}px^{-m}$ ($p \in P$, $r \in R$) and observe that $P \ni px^n = x^n p'$ (for some $p' \in R$) as x is normal, moreover $Rx^n p' = x^n R p' \subseteq P$, $x \notin P$ so we must have $p' \in P$ as P is prime. Thus $rx^{-n}px^{-m} = rp'x^{-(n+m)} \in P\mathbb{X}^{-1}$ so $P\mathbb{X}^{-1}$ is also a left ideal.

Consider $P\mathbb{X}^{-1} \cap R$ this is certainly $\{q \in R : qx^n \in P \text{ for some } n \in \mathbb{N}\}$ but $qx^n R = qRx^n \subseteq P \Rightarrow q \in P$ since $x \notin P$ thus $P\mathbb{X}^{-1} \cap R = P$.

We now show that $P\mathbb{X}^{-1}$ is prime. Let $tR\mathbb{X}^{-1}s \subseteq P\mathbb{X}^{-1}$ for some $t, s \in R\mathbb{X}^{-1}$. Write $s = s'x^{-m}$ and $t = x^{-n}t'$ (we can do this since there is a $t'' \in R$ such that $t = t''x^{-n}$ and there is a t' such that $x^n t' = t'' x^n$ since x is normal) where $s', t' \in R$. Now

$$x^n tR\mathbb{X}^{-1}sx^m = t'R\mathbb{X}^{-1}s' \subseteq x^n P\mathbb{X}^{-1}x^m \subseteq P\mathbb{X}^{-1}$$

hence

$$t'Rs' \subseteq \left(t'R\mathbb{X}^{-1}s'\right) \cap R \subseteq P\mathbb{X}^{-1} \cap R = P$$

and thus either $t' \in P$ or $s' \in P$ since P is prime hence either $t \in P\mathbb{X}^{-1}$ or $s \in P\mathbb{X}^{-1}$ thus $P\mathbb{X}^{-1}$ is prime. //

4.2.43 Proposition

The prime spectrum of the type VII diffusion algebra is as pictured:

$$\begin{tikzcd}
 & \begin{matrix}(RX_1+\eta)R + RX_3R \\ +R(X_2X_1-1)R\end{matrix} \arrow[d, dash] & \\
 & R(X_2X_1-1)R + RX_3R \arrow[dl, dash] \arrow[dr, dash] & \\
R(X_2X_1-1)R \arrow[dr, dash] & & RX_3R \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

where $\eta \in \mathbb{C} \setminus \{0\}$

Proof:

Observe that we can write R as a skew left polynomial ring over the ring $R' := \mathbb{C} < X_1, X_2 : X_1X_2 = 1 >$. $R \cong [X_3; \alpha]R'$ where $\alpha(X_1) = qX_1$ and $\alpha(X_2) = q^{-1}X_2$.

A non–zero prime either contains or doesn't contain X_3. $\frac{R}{RX_3R} \cong R'$ which is prime and RX_3R is minimal among ideals containing X_3; in view of the consideration of primes not containing X_3 that follows, it turns out to be a height one prime. Let $\mathbb{X} := \{X_3^i\}_{i\in\mathbb{Z}^+}$, X_3 is a normal non–zero divisor so this is a right denominator set so, by the preceding lemma, the only possible primes of R not containing X_3 are contractions of primes of $R\mathbb{X}^{-1} \cong [X_3^{\pm 1}; \alpha]R'$. Recall, there is a unique height one prime of R' generated by $X_2X_1 - 1$, this is an α–prime (see [GW89, Ex. 2ZA]) since $\alpha(X_2X_1 - 1) = X_2X_1 - 1$ it is, moreover, the unique non–zero α–ideal (any other α–ideal either contains a polynomial in X_1, hence a monomial in X_1, hence is trivial or is contained in this ideal thus contains an element of the form $\sum X_2^i(X_2X_1 - 1)X_1^j$ premultiplying by X_1^n, where n is the largest value of i occurring and postmultiplying by X_2^m where m is the largest surviving value of j guarantees such an ideal contains $(X_2X_1 - 1)$) thus there is a unique height one prime of $R\mathbb{X}^{-1}$ (as primes clearly contract to α–primes). This prime is the expansion of the ideal $R(X_2X_1 - 1)R$ and $\frac{R}{R(X_2X_1-1)R} \cong [X; \alpha]\left(\mathbb{C}[X_1^{\pm 1}]\right)$, which is a prime ring (the localisation of the quantum plane at powers of X_1), thus, by the preceding lemma, any other primes not containing X_3 must contain this prime hence $R(X_2X_1 - 1)R$ is the only remaining height one prime.

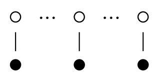
We now show R is a prime ring. It can be graded by X_3–degree. Write the top X_3–degree terms of any two fixed elements, r_1 and r_2, of R as $(p_1(X_1) + I_1)X_3^{n_1}$ and $(p_2(X_1) + I_2)X_3^{n_2}$, respectively, where $I_1, I_2 \in R(X_2X_1 - 1)R$. For sufficiently large value of k, $((p_1(X_1) + I_1)X_3^{n_1})\, X_1^k\, ((p_2(X_1) + I_2)X_3^{n_2}) \neq 0$ so we can never have $r_1Rr_2 = 0$. The result follows. //

4.3 Classification Revisited

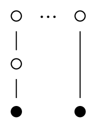
In this section we will address the non-isomorphism of the various types of diffusion algebra we have identified.

4.3.1 Remark

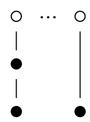
We may reclassify our diffusion algebras by the shape of their prime spectra, as follows:



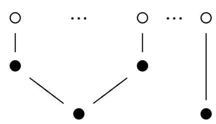
Types I(a), IV(a) and IV(b); (4.3.1.1a)



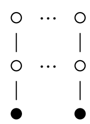
Types I(b) and I(d); (4.3.1.1b)



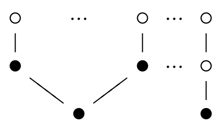
Types II(a),III(e),III(f),IV(f),IV(h) and V(b); (4.3.1.1c)



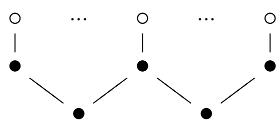
Types I(c),I(e),III(g) and V(a); (4.3.1.1d)



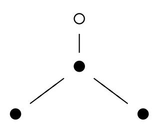
Type I(f); (4.3.1.1e)



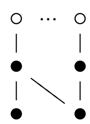
Type I(g); (4.3.1.1f)



Types I(h) and VI; (4.3.1.1g)

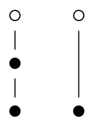


Types II(b) and IV(g); (4.3.1.1h)

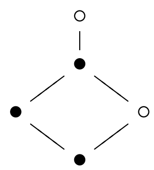


Type II(c); (4.3.1.1i)

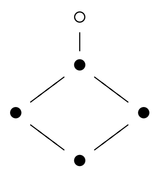
Types III(a) and III(c); (4.3.1.1j)



Types III(b) and III(d); (4.3.1.1k)



Types IV(c),IV(d) and IV(e); (4.3.1.1l)



Type VII; (4.3.1.1m)

(4.3.1.1)

where circles denote infinite families of primes and bullet points denote singleton primes.

4.3.2 Proposition

The algebras of type (4.3.1.1a) *are pairwise non–isomorphic.*

Proof:

For a type I(a) diffusion algebra, there is a maximal ideal containing all three minimal primes; for the type IV(a) and IV(b) diffusion algebras this cannot be the case.

In both the type IV(a) and type IV(b) algebras, there is a unique pair of minimal primes for which no maximal ideal contains them both. Thus any isomorphism must send this pair of one to the pair in the other. For the type IV(a) algebra, this pair is the two ideals $RX_1R + RX_3R$ and $RX_2R + RX_3R$, note that $\mathrm{lann}(RX_2R + RX_3R) \supseteq RX_1R$; for the type IV(b) algebra, this pair is the two ideals RX_1R+RX_2R and RX_1R+RX_3R both of which contain X_1 and hence have zero left annihilators (as X_1 occurs on the extreme right of the PBW–basis). //

4.3.3 Proposition

The two algebras of type (4.3.1.1b) *are non–isomorphic.*

Proof:

In each algebra there is a unique minimal prime not contained in any chain of primes of length 2. For a type I(b) algebra, this ideal is $RX_2R + RX_3R$ which has $\mathrm{lann}(RX_2R + RX_3R) \supseteq RX_1R$; for a type I(d) algebra, this is $RX_1R + RX_3R$ which contains X_1 and hence has zero left annihilator. //

4.3.4 Proposition

The algebras of type (4.3.1.1c) *are pairwise non–isomorphic.*

Proof:

Consider first the factor ring modulo the unique minimal prime that is contained in a chain of primes of length 2.

For a type II(a) or III(e) algebra, this factor ring is isomorphic to the universal envoloping algebra of the 2–d soluble lie algebra. These two algebras can be distinguished by the left annihilators of the other minimal prime, which, for a type II(a) algebra, contains RX_1R; for a type III(e) algebra it is trivial (since X_1 is in the minimal prime).

For a type III(f) or V(b) algebra, this factor ring is isomorphic to a quantised Weyl algebra. These two algebras can again be distinguished by the left annihilators of the other minimal prime in exactly the same way.

Finally, for a type IV(f) or IV(h) algebra, this factor ring is isomorphic to the algebra $\mathbb{C}\langle x, y : xy = 1\rangle$. These two algebras can be distinguished by the right annihilators of the other minimal prime which, for a type IV(h) algebra, contains RX_3R; for a type IV(f) algebra it is trivial (since X_3 is in the minimal prime). //

4.3.5 Proposition

The algebras of type (4.3.1.1d) *are pairwise non–isomorphic.*

Proof:

First consider the left annihilator of the unique minimal prime that is not contained in a chain of primes of length 2. For a type I(c) or V(a) this contains RX_1R; for a type I(e) or III(g) algebra, the annihilator is trivial (since the minimal prime contains X_1).

We distinguish I(c) from V(a) by considering the subgroup, G, of automorphisms of R', the factor ring modulo the minimal prime that is contained in a chain of length two, that are induced by automorphisms of R. Observe that $R' \cong \mathbb{C}\left\langle X_2, X_3 : [X_2, X_3]_q = 0\right\rangle$ for a type I(c) algebra and $R' \cong \mathbb{C}\left\langle X_2, (X_3 - \frac{1}{1-q}) : \left[X_2, \left(X_3 - \frac{1}{1-q}\right)\right]_q = 0\right\rangle$ for a type V(a) algebra. Using [OP95, Prop 3.2], we see that, for a type I(c) algebra, $\alpha \in \mathrm{Aut}(R')$ if and only if $\alpha(X_2^i X_3^j) = \gamma_2^i \gamma_3^j X_2^i X_3^j$ for some nonzero scalars $\gamma_2, \gamma_3 \in \mathbb{C} \setminus \{0\}$. All such automorphism can be induced by an automorphism of R, thus $G \cong (\mathbb{C} \setminus \{0\})^2$. Similarly, for a type V(a) algebra, $\alpha \in \mathrm{Aut}(R')$ if and only if $\alpha\left(X_2^i \left(X_3 - \frac{1}{1-q}\right)^j\right) = \gamma_2^i \gamma_3^j X_2^i \left(X_3 - \frac{1}{1-q}\right)^j$ for some nonzero scalars $\gamma_2, \gamma_3 \in \mathbb{C}\setminus\{0\}$. However, if $\alpha\left(X_3 - \frac{1}{1-q}\right) = \gamma_3\left(X_3 - \frac{1}{1-q}\right)$ for $\gamma_3 \in \mathbb{C} \setminus \{1, 0\}$, α cannot be induced by an automorphism of R (since $X_1\left(X_3 - \frac{1}{1-q}\right) = -\frac{X_1}{1-q}$). Thus $G \cong \mathbb{C} \setminus \{0\}$.

We can distinguish I(e) from III(g) by exactly the same method. The result follows.

//

4.3.6 Proposition

The two algebras of type (4.3.1.1g) *are non–isomorphic.*

Proof:

For a type I(h) algebra, there is a maximal ideal containing all three height one primes; for a type VI algebra, no such maximal can exist. //

4.3.7 Proposition

The two algebras of type (4.3.1.1h) *are non–isomorphic.*

Proof:

Consider the factor ring modulo the unique height one prime. For a type II(b) algebra, this is a commutative polynomial ring over $\mathbb{C}$ whereas, for a type IV(g) algebra this is a localisation of the commutative polynomial ring over $\mathbb{C}$ at two different monic linear polynomials. //

4.3.8 Proposition

The two algebras of type (4.3.1.1j) *are non–isomorphic.*

Proof:

Consider the factor ring modulo the height–3 prime. For a type III(a) algebra, this is a commutative polynomial ring over $\mathbb{C}$ whereas, for a type III(c) algebra this is a Laurent polynomial ring over $\mathbb{C}$. //

4.3.9 Proposition

The two algebras of type (4.3.1.1k) *are non–isomorphic.*

Proof:

Consider the factor ring modulo the unique non–maximal height–1 prime. For a type III(b) algebra, this is a commutative polynomial ring over $\mathbb{C}$ whereas, for a type III(d) algebra this is a Laurent polynomial ring over $\mathbb{C}$. //

4.3.10 Proposition

The algebras of type (4.3.1.1l) *are pairwise non–isomorphic.*

Proof:

Let R and R' be two algebras of type (4.3.1.1l). Present them as in (4.2.30.1) (denote the generators of R' by X_1', X_2' and X_3' and its parameters κ_1' and κ_3'). Assume $\kappa_1 = 0$, $\kappa_1' \neq 0$ and assume that $\Phi : R \longrightarrow R'$ is an isomorphism. If R is graded nontrivially then Φ induces a nontrivial grading on R'. Observe that there is only one nontrivial grading (upto scalar multiplication) on R (resp. R') given by $\deg X_1 = \deg X_3 = 0$, $\deg X_2 = 1$ (resp. $\deg X_1' = \deg X_3' = 0$, $\deg X_2' = 1$).

It follows that $\Phi(X_1) \in \mathbb{C}[X_1', X_3']$. Moreover, $\Phi(X_1)$ is a unit modulo the ideal $I := \mathbb{C}[X_1', X_3'](X_3'X_1' - 1)\mathbb{C}[X_1', X_3']$. Therefore, either $\Phi(X_1) = \gamma {X_1'}^i + g$ or $\Phi(X_1) = \gamma {X_3'}^i + g$, $\gamma \in \mathbb{C} \setminus \{0\}$, $g \in I$. Similarly, $\Phi(X_3) = \gamma^{-1} {X_3'}^i + g'$ or $\Phi(X_3) = \gamma^{-1} {X_1'}^i + g'$, $g' \in I$. Since $\Phi(X_1 X_3) = 1$, we must have $\Phi(X_1) = \gamma {X_1'}^i + g$. We shall assume that $\Phi\mid_{\mathbb{C}[X_1, X_3]}$ surjects onto $\mathbb{C}[X_1', X_3']$ (otherwise we would have a contradiction).

We also deduce that $\Phi(X_2) \in \mathbb{C}[X_1', X_3']X_2'\mathbb{C}[X_1', X_3']$, write $\Phi(X_2) = \sum_i P_i(X_3')X_2'{X_1'}^i$. Since Φ is an isomorphism, $\Phi(X_1X_2) = 0 \Rightarrow {X_1'}^N \Phi(X_1)\Phi(X_2) = 0 \Rightarrow {X_1'}^{N+i}\Phi(X_2) = 0$ since we may choose N sufficiently large to kill g. We may, moreover, deduce that $X_1'\Phi(X_2) = 0$ since $\operatorname{lann}(X_2) = RX_1$, thus there are no elements in R which don't left–annihilate X_2 but have nth powers that do, thus there are no such elements in $\Phi(R) \ni X_1'$ (this time left annihilating $\Phi(X_2)$, of course). If $X_1' \sum_i P_i(X_3')X_2'{X_1'}^i = 0$, then $P_i(X_3') = \gamma_i(\kappa_1' X_3' - 1)$ for all i, where $\gamma_i \in \mathbb{C}$. Hence $X_2' \notin \mathbb{C}[X_1', X_3']\Phi(X_2)\mathbb{C}[X_1', X_3']$ since

$$ {X_3'}^{i_3}{X_1'}^{i_1}\Phi(X_2) = \begin{cases} 0, & i_1 \geqslant 1 \\ \sum_i {X_3'}^{i_3} P_i(X_3')X_2', & \text{otherwise,} \end{cases} $$

contradicting surjectivity of Φ.

One can distinguish $\kappa_3 = 0$ form $\kappa' \neq 0$ in exactly the same way but considering $\Phi(X_2X_3)$. //

We summarise these propositions in the following theorem:

4.3.11 Theorem

In (4.1.3)*, the various types of diffusion algebra distinguished are all pairwise non–isomorphic.*

//

Finally, we shall examine a 4–generator diffusion algebra that exhibits the property of containing two non–isomorphic 3–generator diffusion algebras. This is in contrast to the noetherian situation where the two noetherian 3–generator diffusion algebras (other than multiparameter quantum affine 3–space) cannot simultaneously occur in a noetherian n–generator diffusion algebra.

4.3.12 Lemma

The algebra generated by x_1, x_2, x_3 *and* x_4 *subject to the relations:*

$$
\begin{aligned}
(q_1+1)(x_1x_2 - q_1x_2x_1) &= -x_2 && (4.3.12.1a)\\
(q_2+1)(x_1x_3 - q_2x_3x_1) &= -x_3 && (4.3.12.1b)\\
x_1x_4 &= x_1 - x_4 && (4.3.12.1c)\\
x_2x_3 &= 0 && (4.3.12.1d)\\
(q_1+1)(x_2x_4 - q_1x_4x_2) &= x_2 && (4.3.12.1e)\\
(q_2+1)(x_3x_4 - q_2x_4x_3) &= x_3 && (4.3.12.1f)
\end{aligned}
\tag{4.3.12.1}
$$

is a diffusion algebra.

Proof:

It is clear that the relations are of the form (2.1.1.1). After the diamond lemma, it suffices to check associativity for the words $x_1x_2x_3, x_1x_2x_4, x_1x_3x_4$ and $x_2x_3x_4$. We show the reduction of $x_1x_2x_4$ as an example (the others are similar):

$$\begin{aligned}
(x_1x_2)x_4 &= \left(\frac{q_1}{q_1+1}x_2x_1 - \frac{1}{q_1+1}x_2\right)x_4 \\
&= \frac{q_1}{q_1+1}x_2(x_1-x_4) - \frac{1}{q_1+1}x_2x_4 \\
&= \frac{q_1}{q_1+1}x_2x_1 - \frac{q_1}{q_1+1}x_4x_2 - \frac{1}{q_1+1}x_2 \text{ and} \\
x_1(x_2x_4) &= x_1\left(\frac{q_1}{q_1+1}x_4x_2 + \frac{1}{q_1+1}x_2\right) \\
&= \frac{q_1}{q_1+1}(x_1-x_4)x_2 + \frac{1}{q_1+1}x_1x_2 \\
&= \frac{q_1}{q_1+1}x_2x_1 - \frac{q_1}{q_1+1}x_4x_2 - \frac{1}{q_1+1}x_2.
\end{aligned}$$

//

4.3.13 Proposition

The diffusion algebra (4.3.12.1) *has subalgebras both of type VI and type VII.*

Proof:

From (4.1.3), we deduce that the subalgebras $\mathbb{C}[x_1, x_2, x_3]$ and $\mathbb{C}[x_2, x_3, x_4]$ are both of type VI. Whereas, the subalgebras $\mathbb{C}[x_1, x_2, x_4]$ and $\mathbb{C}[x_1, x_3, x_4]$ are both of type VII.

//

4.3.14 Remark

In the previous chapter, we discovered that any noetherian diffusion algebra in arbitrarily many generators could be constructed from one of the three types of 2–generator noetherian diffusion algebra by adding new generators that q–commute with each other and the generators of the 2–generator subalgebra in a compatible way. The preceding proposition suggests that, for non–noetherian diffusion algebras, the situation is quite different, mixed behaviour can occur suggesting that classifying non–noetherian diffusion algebras in arbitrarily many generators is an intractible problem.

4.4 Primitive Ideals

In this section, we shall assess the primitivity of the various prime ideals of non–noetherian diffusion algebras. First we recall the primitive spectrum of the two 2–generator diffusion algebras.

The ring $\mathbb{C}\langle x, y : xy = 1\rangle$ was studied in [Jor00]. It is a primitive ring, the unique height one prime is not primitive, the maximals, of course, are.

The ring $\mathbb{C}\langle x, y : xy = 0\rangle$ is easily examined. The factor ring modulo either of the minimal primes is a polynomial ring in one variable, which is not primitive. The maximals, as always, are primitive.

We shall assume familiarity with the primitive spectra of these two and of the three noetherian 2–generator diffusion algebras in the following section. The primitive ideals of a large number of diffusion algebras are now easily identified.

4.4.1 Proposition

The Primitive spectra of all non–noetherian diffusion algebras aside from types III(a), III(b), III(c), III(d), IV(c), IV(d), IV(e), IV(g) and VII are listed below:

- *Type I:*
  - *Type I(a): maximals only.*
  - *Type I(b): maximals only.*
  - *Type I(c): RX_1R and all maximals.*
  - *Type I(d): maximals only.*
  - *Type I(e): RX_2R and all maximals.*
  - *Type I(f): maximals only.*
  - *Type I(g): RX_1R and all maximals.*
  - *Type I(h): RX_1R, RX_2R and all maximals.*
- *Type II:*

  - *Type II(a): RX_1R and all maximals.*
  - *Type II(b): RX_1R, RX_2R and all maximals.*
  - *Type II(c): RX_1R, RX_2R and all maximals.*

- *Type III:*
  - *Type III(e): RX_2R and all maximals.*
  - *Type III(f): RX_2R and all maximals.*
  - *Type III(g): RX_2R and all maximals.*

- *Type IV:*
  - *Type IV(a): maximals only.*
  - *Type IV(b): maximals only.*
  - *Type IV(f): RX_2R and all maximals.*
  - *Type IV(h): RX_3R and all maximals.*

- *Type V:*
  - *Type V(a): RX_1R and all maximals.*
  - *Type V(b): RX_1R and all maximals.*

- *Type VI: RX_1R, RX_2R and all maximals.*

Proof:

We already understand the prime spectrum of all these rings. Primitive ideals are prime so we need only assess the factor rings by each of the prime ideals. These are always diffusion algebras in fewer numbers of generators, for which we already understand primitivity. //

4.4.2 Proposition

Let R be a type III(a) or a type III(c) diffusion algebra, then the primitive ideals are $RX_2(X_2 - \eta)R$ where $\eta \in \mathbb{C} \setminus \{0\}$, RX_2R and all maximals.

Proof:

Suppose that M is a simple (left) module for R. If $X_2M = 0$ then M is not faithful. Otherwise there exists $m \in M$ such that $X_2m \neq 0$. Then $RX_2m = M$ (by simplicity) thus $X_2RX_2m = X_2M$ thus there is an $r \in R$ such that $X_2rX_2m = X_2m$. Since $X_2RX_2 = X_2^2\mathbb{C}[X_2]$, there exists $f(X_2) \in \mathbb{C}[X_2] \setminus \{0\}$ such that $X_2f(X_2)m = 0$. Then $X_2f(X_2)M = X_2f(X_2)RX_2m = \mathbb{C}[X_2]X_2^2f(X_2)m = 0$. So again M is not faithful thus R is not primitive.

Let $R' := \frac{R}{RX_2(X_2-\eta)R}$. Then $R(X_2-\eta)$ is a maximal left ideal (for $\eta \neq 0$). Observe that $X_1 \in R(X_2-\eta)$ so any bigger right ideal J contains a polynomial, P, in X_3. But, again since $X_1 \in J$, $\mathbb{C}[X_3, X_1]P\mathbb{C}[X_3, X_1] \subseteq J$, we have $((q-1)X_3X_1 + X_3 + X_1 + \kappa)^n \in J$ for some $n \in \mathbb{N}$ (as any ideal in the quantised Weyl algebra contains a power of this *casimir* element), thus $X_2\left((q-1)X_3X_1 + X_3 + X_1 + \kappa\right)^n \in J$, which is congruent modulo X_1 to $\left(\prod_{i=1}^{n}(\kappa + [i-1]_q)\right) X_2$ (using the proof of (4.2.12)) so J is trivial. Now, X_2 acts non–trivially on the left module $\frac{R}{R(X_2-\eta)}$ thus $X_2 \notin \operatorname{ann}\left(\frac{R}{R(X_2-\eta)}\right)$. But every nonzero prime in R' contains X_2 so $\operatorname{ann}\left(\frac{R}{R(X_2-\eta)}\right) = 0$ and R' is primitive.

The ideal RX_2R is primitive since its factor ring, in either case, is a primitive 2–generator diffusion algebra. //

4.4.3 Proposition

Let R be a diffusion algebra of type III(b) or III(d), then the primitive ideals are RX_2R and all maximals.

Proof:

It suffices to show that the ideal $RP_{n+1}(X_3)R$ is not primitive. Any polynomial in X_2 is incongruent to zero modulo this ideal as we know from the representations ρ_η (defined in (4.2.23)) that $X_2 - \eta \notin RP_{n+1}(X_3)R$ for all $\eta \in \mathbb{C} \setminus \{0\}$. We can now use exactly the same method as in the first paragraph of the proof of the preceding proposition to prove that $\frac{R}{RP_{n+1}(X_3)R}$ is not primitive. //

4.4.4 Proposition

Let R be a type IV(c), IV(d) or IV(e) diffusion algebra, then the primitive ideals of R are RX_2R, $RX_2(X_2-\eta)R + Rf_\eta R$ and all maximals.

Proof:

Again, the methods of the first paragraph of the proof of (4.4.2) show that R is not a primitive ring. It is clear that RX_2R is a primitive ideal. We now examine the factor ring $R' := \frac{R}{RX_2(X_2-\eta)R+Rf_\eta R}$. If $X_1X_2 = 0$ then $J := RX_1 + R(X_2-\eta)$ is a non–trivial left ideal, otherwise $X_1X_2 = X_2$ and $J := R(X_1-1) + R(X_2-\eta)$ is non–trivial (in the first case, $X_1(X_2-\eta) \in RX_1 \subseteq J$; in the second, $X_1(X_2-\eta) = X_2 - X_1\eta = (X_2-\eta) - \eta(X_1-1) \in J$). In either case J is maximal since any bigger left ideal, J', must contain a polynomial in X_3. If necessary, premultiplying by X_1^i guarantees that there is a polynomial $p(X_3) \in J'$ with constant term $p_0 \neq 0$. Premultiplying by $X_3X_1 - 1$ implies that J' contains $p_0(X_3X_1-1)$ and since $J' \supseteq J \ni X_1$, $p_0 \in J'$ so J' is trivial. Now, X_2 acts non–trivially on the left module $\frac{R'}{J}$ thus $X_2 \notin \operatorname{ann}\left(\frac{R'}{J}\right)$ but any nonzero prime of R' contains X_2 thus $\operatorname{ann}\left(\frac{R'}{J}\right) = 0$. //

4.4.5 Proposition

Let R be a type IV(g) diffusion algebra, then the primitive ideals of R are the ideals $R\left(1-(X_2-1)X_1\right)R$, $R\left(1-X_3(X_2+\lambda)\right)R$ and all maximals.

Proof:

We may view $R' := \frac{R}{R(1-X_3(X_2+\lambda))R}$ as the localisation of the 2–generator diffusion algebra $S := \mathbb{C}\left\langle X_1, (X_2-1) : X_1(X_2-1) = 1\right\rangle$ at the denominator set, $\mathbb{X}$, of powers of $(X_2+\lambda)$. The left ideal $R'X_1$ is maximal and $\operatorname{ann}\left(\frac{R'}{R'X_1}\right) = 0$. The other ideal is treated similarly. //

4.4.6 Proposition

Let R be a type VII diffusion algebra, then the primitive ideals are 0, $R(X_2X_1-1)R$, RX_3R *and all maximals.*

Proof:

It is enough to look at 0 as the factor rings by the others are understood. Consider $RX_1+R(X_3+1)$, this is a maximal left ideal (any bigger left ideal, J, contains a polynomial in X_2, without loss of generality monic of degree n, premultiply by X_1^n and reduce modulo RX_1 top deduce that $1 \in J$) on which both X_3 and X_2X_1-1 act non–trivially thus the left module given by factoring out this left ideal must be a simple faithful module. $/\!\!/$

4.4.7 Remark

We observe that the primitive spectra of non–noetherian diffusion algebras yielded no surprises: a height n prime is primitive if and only if the height $n+1$ spectrum is finite.

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