Reversible skew Laurent polynomial rings, rings of invariants and related rings

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To my parents - Banjong Sasom and Samran Sasom

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Summary

The thesis begins by answering the question of when a skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ has an automorphism θ of order two transposing x and x^{-1} and restricting to an automorphism of R. When such an automorphism exists, we call the ring S reversible and refer to θ as a reversing automorphism. The two best known examples of simple skew Laurent polynomial rings of Gelfand Kirillov dimension two are the localization of the enveloping algebra $\mathbb{C}[x, y : xy - yx = x]$ at the powers of x and the coordinate ring of the quantum torus $\mathbb{C}[x^{\pm 1}, y^{\pm 1} : xy = qyx]$ when q is not a root of unity. We show that each of these is reversible and compute the rings of invariants for the reversing automorphisms in terms of generators and relations. In each case, we present the ring of invariants in the form T/pT where T is an algebra with three generators and three relations and p is a central element. For the localized enveloping algebra, T is an iterated skew polynomial ring, involving both automorphisms and derivations, but this does not appear to be the case for the quantum torus, where we denote T by T_q. The last two chapters of the thesis are concerned with this algebra T_q which can be regarded as a quantization of the enveloping algebra $U(sl_2)$ or of the commutative polynomial ring A in three indeterminates and has featured in the literature of mathematical physics and linear algebra. We classify the finite-dimensional simple modules over T_q showing that there are five of each dimension, in contrast to the well known situation for $U(sl_2)$ which has one of each. We give a geometrical explanation of the role of the number five by showing that T_q induces the structure of a Poisson algebra on A in such a way that there are five Poisson maximal ideals and that, for each of these, there is one d-dimensional simple Poisson module for each positive integer d.

Contents

Introduction

In this thesis, we consider skew Laurent polynomial rings $S = R[x^{\pm 1}; \alpha]$ and ask when there is an automorphism θ of S such that $\theta(x) = x^{-1}$ and θ restricts to an automorphism γ of R such that $\gamma^2 = \mathrm{id}_R$. If such an automorphism θ exists, we call S reversible and θ a reversing automorphism. Let $\gamma \in \mathrm{Aut}(R)$ be such that $\gamma^2 = \mathrm{id}_R$. We say that α is $\gamma-$reversible if $\gamma\alpha\gamma^{-1} = \alpha^{-1}$. We then show that S is reversible if and only if α is $\gamma-$reversible for some $\gamma \in \mathrm{Aut}\, R$ such that $\gamma^2 = \mathrm{id}_R$.

The two most familiar examples of skew Laurent polynomial rings turn out to be reversible and we compute the fixed ring for the reversing automorphism in each case.

For the first, let $R = \mathbb{F}[t]$ where $\mathbb{F}$ is an algebraically closed field, with $\mathrm{char}\, \mathbb{F} = 0$. Let α and γ be the $\mathbb{F}$-automorphisms such that $\alpha(t) = t + 1$ and $\gamma(t) = -t$. It is easily checked that $\gamma\alpha\gamma^{-1} = \alpha^{-1}$ so α is $\gamma-$reversible. The skew polynomial ring $R[x; \alpha]$ is generated by two elements x and t subject to the relation

$$xt - tx = x.$$

This is the enveloping algebra of the two-dimensional non-abelian solvable Lie algebra and the skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ is its localization at the powers of the normal element x. It is well-known that S is simple, see [8, p. 23].

In the localized enveloping algebra $S = \mathbb{F}[t, x^{\pm 1} : xt - tx = x]$, let θ be the reversing automorphism of S. Thus $\theta(t) = -t$ and $\theta(x) = x^{-1}$. We shall show that the ring of invariants S^θ is generated by three generators $u = t^2$, $v = x + x^{-1}$ and $w = tx - tx^{-1}$ and identify S^θ as a homomorphic image T/pT where T is an iterated skew polynomial ring over $\mathbb{F}$ in three generators U, V, W and p is a central element of T with total degree 3;

$$p = (4 - V^2)U + W^2 + 3VW + V^2 + 4.$$

The main tools used here are [11, Proposition 1], which shows that T/pT is a domain, and GK dimension which then show that $T/pT \simeq S^\theta$.

For the second ring, let $R = \mathbb{F}[t^{\pm 1}]$ be an ordinary Laurent polynomial ring over any arbitrary field $\mathbb{F}$. Let α and γ be the $\mathbb{F}$-automorphisms such that $\alpha(t) = qt$, where $q \in \mathbb{F}\backslash\{0\}$ and $\gamma(t) = t^{-1}$. It is easily checked that $\gamma\alpha\gamma^{-1} = \alpha^{-1}$ so α is $\gamma-$reversible. The skew polynomial ring $S = R[x^{\pm 1}; \alpha]$ is sometimes called the quantum torus, see [8, p. 16]. It is known that S is simple if and only if q is not a root of unity, see [8, p. 23].

Here $S = \mathcal{O}_q((\mathbb{F}^\times)^2) = \mathbb{F}[t^{\pm 1}, x^{\pm 1} : xt = qtx]$ and the reversing automorphism θ of S is such that $\theta(t) = t^{-1}$ and $\theta(x) = x^{-1}$. We shall show that, if q is not a root of unity, the ring of invariants S^θ is generated by three generators $u = t + t^{-1}$, $v = x + x^{-1}$ and $w = tx + t^{-1}x^{-1}$ and identify S^θ as a homomorphic image T_q/pT_q where T_q is an $\mathbb{F}-$algebra, with $GK\dim(T_q) = 3$, generated by three generators U, V, W subject to the relations

$$\begin{aligned} UV - qVU &= (1 - q^2)W \\ VW - qWV &= q^{-1}(1 - q^2)U \\ WU - qUW &= q^{-1}(1 - q^2)V. \end{aligned}$$

and p is a central element of T_q with total degree 3;

$$p = WVU - qW^2 - q^{-1}V^2 - U^2 + 2(1 + q^{-2}).$$

If $q^2 \neq 1$ then we shall change the generators to X, Y, Z and these three relations become

$$XY - qYX = Z, \quad YZ - qZY = X, \quad ZX - qXZ = Y, \tag{1}$$

where $X = \frac{q^{1/2}}{1-q^2}U$, $\quad Y = \frac{q^{1/2}}{1-q^2}V$, and $Z = \frac{q}{1-q^2}W$.

Once again [11, Proposition 1] and GK dimension are used but [11, Proposition 1] is not applied directly as there is no obvious way of writing T_q as an iterated skew polynomial ring. However we find a filtration of T_q such that the associated graded ring is an iterated skew polynomial ring to which [11, Proposition 1] can be applied. Another useful tool here is Bergman's Diamond Lemma [2] which we use to find appropriate bases for the algebras being studied. Our result on S^θ extends a known result in the commutative case. If $S = \mathbb{F}[x^{\pm 1}, y^{\pm 1}]$ and θ is the $\mathbb{F}-$automorphism such that $\theta(x) = x^{-1}$ and $\theta(y) = y^{-1}$ then it is known that S^θ is generated by commuting elements $u = t + t^{-1}, v = x + x^{-1}$ and $w = tx + t^{-1}x^{-1}$ subject to the relation $wvu = u^2 + v^2 + w^2 - 4$. See [18, Example 3.5], although there the author works over $\mathbb{Z}$ rather than a field.

The above results are in Chapter 2. In Chapter 3, we study the ring T_q that occurred in Chapter 2. If q is set to 1 then T_q becomes the enveloping algebra $U(so_3)$, which is isomorphic to $U(sl_2)$, but it can also be viewed as a quantization of the commutative polynomial ring in three indeterminates.

Our main aim is to classify the finite-dimensional simple T_q-modules. We shall identify a one-parameter family of Verma-like modules $V(\eta)$ such that every finite-dimensional simple module is isomorphic to a factor of $V(\eta)$ for some $\eta \in \mathbb{F}$. For the action of x the module $V(\eta)$ has a countable basis of eigenvectors and generalized eigenvectors with eigenvalues in a sequence $\lambda_1, \lambda_2, \lambda_3, \dots$ determined by η.

The nature of the sequence $\{\lambda_i\}_{i\geq 1}$, denoted by $S(\eta)$, splits into three cases:

Case I, all distinct : the terms of $S(\eta)$ are distinct; $\eta^2 \neq q^{2-n}$ for all $n > 1$.

Case II, even repeating : $\eta^2 = q^{2-2d}$ for some positive integer d;

$\lambda_1 = \lambda_{2d}, \lambda_2 = \lambda_{2d-1}, \dots, \lambda_d = \lambda_{d+1}$; otherwise the terms of $S(\eta)$ are distinct.

Case III, odd repeating : $\eta^2 = q^{2-2d-1}$ for some positive integer d;

$\lambda_1 = \lambda_{2d+1}, \lambda_2 = \lambda_{2d}, \dots, \lambda_d = \lambda_{d+2}$; otherwise the terms of $S(\eta)$ are distinct.

In Case I, we shall show that $V(\eta)$ has length ≤ 2 and determine when there is a $d-$dimensional simple factor. For each $d \geq 1$, there are two values of η for which there is a d-dimensional simple factor but the two factors are isomorphic. So this case gives one $d-$dimensional simple module for each d. In Case II, $V(\eta)$ has length ≤ 3. For each d, there are two values of η for which $V(\eta)$ has a $2d-$dimensional factor, which splits into a direct sum of two $d-$dimensional simple modules. This gives four $d-$dimensional simple modules. Finally, in Case III, which is technically the hardest, we shall see that $V(\eta)$ is always simple. To summarise, there are five $d-$dimensional simple T_q-modules for each $d \geq 1$ when q is not a root of unity. If we set $q = 1$ in (1) then T_q becomes $U(sl_2)$. It is well-known that there is a unique $d-$dimensional simple $U(sl_2)$-module for each $d \geq 1$, see [19].

We show that each finite-dimensional simple left T_q-module gives rise to a Leonard triple, in the sense of Terwilliger [22].

In Chapter 4, we study a Poisson algebra related to T_q. In this sort of situation there is often a "semi-classical case" namely a Poisson algebra associated to T_q.

Some authors might call T_q a quantum or quantized algebra, and refer to T_1 as the classical case and the Poisson algebra as the semi-classical case. So as the quantum algebra has five $d-$dimensional simple modules and the classical case has only one for each dimension d, it is natural to ask about the simple finite-dimensional Poisson modules for the semi-classical case.

To construct this Poisson algebra, we use the method described in [3, III.5] but modify the generators so that the relations become

$$xy - qyx = (q-1)z, \quad yz - qzy = (q-1)x, \quad zx - qxz = (q-1)y.$$

This algebra is isomorphic to the previous version when $q \neq 1$ but when $q = 1$ it is a commutative polynomial algebra. We next replace q by an indeterminate. Thus we consider the algebra T generated by $x,\ y,\ z,\ t,\ t^{-1}$ subject to the relations

$$xy - tyx = (t-1)z, \quad yz - tzy = (t-1)x, \quad zx - txz = (t-1)y,$$

and

$$xt = tx, \quad yt = ty, \quad zt = tz, \quad tt^{-1} = 1 = t^{-1}t.$$

In this algebra, t is a central non-unit non-zero-divisor such that $T/(t-1)T$ is commutative and isomorphic to $A := \mathbb{C}[x, y, z]$. This induces a Poisson bracket $\{-,-\}$ on A such that, for $\alpha,\ \beta \in T$,

$$\{\overline{\alpha}, \overline{\beta}\} = \overline{(t-1)^{-1}[\alpha, \beta]}.$$

This satisfies the equations,

$$\{x, y\} = yx + z, \quad \{y, z\} = zy + x, \quad \{z, x\} = xz + y.$$

Our aim is to classify the finite-dimensional modules and to see whether they confirm to the classical pattern, with one of each dimension, or the quantum pattern, with five of each dimension, or something else. We will see that the annihilator of a finite-dimensional simple Poisson module is both a Poisson ideal and a maximal ideal so we begin by determining the Poisson maximal ideals. We find that there are five Poisson maximal ideals J_i, $1 \leq i \leq 5$, of A for the Poisson bracket:

$$\begin{aligned} J_1 &= xA + yA + zA, \\ J_2 &= (x+1)A + (y+1)A + (z+1)A, \\ J_3 &= (x+1)A + (y-1)A + (z-1)A, \\ J_4 &= (x-1)A + (y+1)A + (z-1)A \quad \text{and} \\ J_5 &= (x-1)A + (y-1)A + (z+1)A. \end{aligned}$$

Next, we will classify finite-dimensional simple Poisson $A-$modules annihilated by J_i for $1 \leq i \leq 5$. We shall see that, for $d \geq 1$, the Poisson algebra A has precisely one $d-$dimensional simple Poisson module annihilated by each J_i and therefore has precisely five $d-$dimensional simple Poisson modules.

In Section 4.5, we shall calculate a Poisson algebra arising from the quantized enveloping algebra $U_q(sl_2)$, using a presentation discovered by Ito, Terwilliger and Weng [23]. We then classify the finite-dimensional simple Poisson modules for this Poisson algebra. Let $q \neq 1$. The quantized enveloping algebra $U_q(sl_2)$ has a presentation, established in [23], with generators $x^{\pm 1}$, y, z and relations $xx^{-1} = x^{-1}x = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

Multiplying through by $q-q^{-1}$ and replacing q by an indeterminate we obtain the $\mathbb{C}-$algebra T with generators $x,\ y,\ z$ and $t^{\pm1}$ subject to the relations

$$xy-t^{-2}yx=1-t^{-2},\quad yz-t^{-2}zy=1-t^{-2},\quad zx-t^{-2}xz=1-t^{-2}$$

and

$$xt=tx,\quad yt=ty,\quad zt=tz,\quad tt^{-1}=1=t^{-1}t.$$

Again $A:=T/(t-1)T\simeq\mathbb{C}[x,y,z]$ is a commutative polynomial algebra and this time the Poisson bracket is given by

$$\{x,y\}=2(1-yx),\quad \{y,z\}=2(1-zy),\quad \{z,x\}=2(1-xz).$$

There are two Poisson maximal ideals of A:

(i) $I_1=(x-1)A+(y-1)A+(z-1)A$;

(ii) $I_2=(x+1)A+(y+1)A+(z+1)A$.

We shall see that, for $d\geq 1$, the Poisson module A has exactly one $d-$dimensional simple Poisson module annihilated by $I_j,\ j=1,2$, and so there are two $d-$dimensional simple Poisson modules for each $d\geq 1$.

The material in Chapter 2, 3 and 4 is original. Chapter 1 contains the preliminary standard material that is applied elsewhere. The localized enveloping algebra and the quantum algebra in Chapter 2 are well-known examples of simple algebras. D. A. Jordan, my supervisor, suggested the problems, how they might be solved and helped with some of the details. The finite-dimensional simple T_q-modules have been classified, using different methods not involving Verma-like modules, by Havlicek, Klimyk and Posta [9] and [10]. The method used in this thesis is independent of these papers as we only become aware of their existence very recently.

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Chapter 1

Preliminaries

This chapter contains some of the background material that will be used throughout this thesis. The main topics are rings of invariants, skew Laurent polynomial rings, GK dimension, associated graded rings, the Diamond Lemma, Leonard triples, Poisson algebras and Poisson modules.

1.1 Notation

Throughout this thesis, $\mathbb{F}$ is an algebraically closed field, with $\operatorname{char}\mathbb{F} \neq 2$, and $\mathbb{F}^{\times}$ is the multiplicative group of non-zero elements in $\mathbb{F}$.

Definition 1.1.1. Let R be any ring. The **centre** of R, denoted $Z(R)$, is the set

$$Z(R) = \{r \in R : rs = sr \quad \text{for all } s \in R\}.$$

Definition 1.1.2. [8, p. 55] Let A be a left module over a ring R. Give any subset $X \subseteq A$, the **annihilator** of X is the set

$$\operatorname{ann}_R(X) = \{r \in R : rx = 0 \quad \text{for all } x \in X\},$$

which is a left ideal of R.

For a ring R, $\operatorname{Aut} R$ denotes the group of automorphisms of R and if R is an $\mathbb{F}-$algebra, $\operatorname{Aut}_{\mathbb{F}} R$ denotes the group of $\mathbb{F}-$automorphisms of R, that is, those automorphisms θ of R such that $\theta(\lambda r) = \lambda\theta(r)$ for all $\lambda \in \mathbb{F}$ and $r \in R$.

1.2 Rings of Invariants

Definition 1.2.1. Let R be any ring and γ be an automorphism of R. The **ring of invariants**, or **fixed ring**, is

$$R^\gamma = \{r \in R \,|\, \gamma(r) = r\}.$$

We need the following results on rings of invariants.

Proposition 1.2.2. *Let γ be an automorphism of a ring R with finite order n and let M be a left $R-$module. Suppose that $n^{-1} \in R$. If M is Noetherian as an $R-$module then it is Noetherian as an $R^\gamma-$module.*

Proof. This is the left-hand version of a special case of [20, Corollary 26.13]. □

Proposition 1.2.3. *Let γ be an automorphism of a ring R with finite order n. If R is a simple domain and $n^{-1} \in R$ then R^γ is a simple ring.*

Proof. As R is a domain, so too is its subring R^γ. If R^γ is not simple then it has a non-zero maximal ideal Q. By [20, Theorem 28.3(ii)], R has a prime ideal T such that Q is a minimal prime over $T \cap R^\gamma$. As 0 is prime and $Q \neq 0$, $T \cap R^\gamma \neq 0$ and therefore $T \neq 0$, contradicting the simplicity of R. □

Here is an example of a ring of invariants.

Example 1.2.4. Let $R = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ and γ be the automorphism of R such that $\gamma(x_1) = x_1^{-1}$ and $\gamma(x_2) = x_2^{-1}$. Let $\beta_i = x_i + x_i^{-1}$, $i = 1, 2$, and let $\theta = x_1 x_2^{-1} + x_1^{-1} x_2$. Then it is clear that β_1, β_2, $\theta \in R^\gamma$ and it is known, see [18, Example 3.5], that they generate R^γ and that

$$R^\gamma \cong \mathbb{Z}[x, y, z]/(x^2 + y^2 + z^2 - xyz - 4)$$

under an isomorphism ϕ such that $\phi(\beta_1) = \overline{x}$, $\phi(\beta_2) = \overline{y}$ and $\phi(\theta) = \overline{z}$. The same is true if we replace $\mathbb{Z}$ with a field.

1.3 Skew Laurent polynomial rings

In this section, we give some results for general skew polynomial rings and skew Laurent polynomial rings.

Definition 1.3.1. [8, p. 26] A **derivation** on a ring R is any map $\delta : R \to R$ satisfying

$$\begin{aligned} \delta(r+s) &= \delta(r)+\delta(s) \quad \text{and} \\ \delta(rs) &= \delta(r)s + r\delta(s) \quad \text{for all } r,\ s \in R. \end{aligned}$$

Note that

$$\delta(1) = \delta(1 \cdot 1) = \delta(1)1 + 1\delta(1) = 2\delta(1),$$

whence we automatically have $\delta(1) = 0$.

Definition 1.3.2. [8, p. 33] Let α be an endomorphism of a ring R. A (left) $\alpha-$**derivation** on R is any additive map $\delta : R \to R$ such that

$$\delta(rs) = \alpha(r)\delta(s) + \delta(r)s \quad \text{for all } r,\ s \in R.$$

If R is an $\mathbb{F}-$algebra and $\alpha \in \mathrm{Aut}_{\mathbb{F}}(R)$ then δ is an $\alpha - \mathbb{F}-$**derivation** if δ is $\mathbb{F}-$linear.

Definition 1.3.3. [8, p. 34] Let R be a ring, α a ring endomorphism of R, and δ an α-derivation on R. We shall write $S = R[x;\alpha,\delta]$ provided

(i) S is a ring, containing R as a subring;

(ii) x is an element of S;

(iii) S is a free left R-module with basis $\{1, x, x^2, x^3, \ldots\}$;

(iv) $xr = \alpha(r)x + \delta(r)$ for all $r \in R$.

Such a ring S is called a **skew polynomial ring** over R.

When α is the identity map on R, we write $S = R[x;\delta]$ and when $\delta = 0$, we write $R[x;\alpha]$.

Definition 1.3.4. [8, p. 15] Let R be a ring and α an automorphism of R. We write $S = R[x^{\pm 1}; \alpha]$ and call R a **skew Laurent polynomial ring**, if

(i) S is a ring, containing R as a subring;

(ii) x is an invertible element of S;

(iii) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$;

(iv) $xr = \alpha(r)x$ for all $r \in R$.

Proposition 1.3.5. *Given a ring R, a ring endomorphism α of R, and an α-derivation δ on R, there exists a skew polynomial ring $R[x; \alpha, \delta]$.*

Proof. See [8, Proposition 2.3]. □

Theorem 1.3.6. *Let $S = R[x; \alpha, \delta]$.*

(i) If σ is injective and R is an integral domain, then S is an integral domain.

(ii) If σ is an automorphism and R is right (left) Noetherian, then S is right (left) Noetherian.

Proof. See [19, Theorem 1.2.9]. □

Proposition 1.3.7. *Let α be an automorphism of a ring R and $T = R[x^{\pm 1}; \alpha]$. If R is right (left) Noetherian, then so is T.*

Proof. See [8, Corollary 1.15]. □

In particular, if R is commutative Noetherian then $R[x]$ is Noetherian. It follows that every commutative finitely generated $\mathbb{F}-$algebra is Noetherian.

Corollary 1.3.8. *Let $\mathbb{F}$ be a field and $q \in \mathbb{F}^\times$. Then $\mathcal{O}_q((\mathbb{F}^\times)^2)$ is a simple ring if and only if q is not a root of unity.*

Proof. See [8, Corollary 1.18]. □

The following result is the universal property for skew polynomial rings.

Lemma 1.3.9. *Let R be a ring, α an automorphism of R, and $S = R[x,\alpha]$. Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi\alpha(r)y$ for all $r \in R$. Then there is a unique ring homomorphism $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$.*

Proof. See [8, Lemma 1.11]. □

We shall need the following universal property for skew Laurent polynomial rings.

Proposition 1.3.10. *[8, Exercise 1N] Let α be an automorphism of a ring R and $T = R[x^{\pm 1};\alpha]$. Suppose that we have a ring U, a ring homomorphism $\phi : R \to U$, and a unit $y \in U$ such that $y\phi(r) = \phi\alpha(r)y$ for all $r \in R$. Then there is a unique ring homomorphism $\psi : T \to U$ such that $\psi|_R = \phi$ and $\psi(x) = y$.*

Proposition 1.3.11. *Let $R = \mathbb{F}[x_1;\alpha_1,\delta_1][x_2;\alpha_2,\delta_2]\cdots[x_n;\alpha_n,\delta_n]$ be an iterated skew polynomial ring over $\mathbb{F}$ where, for $1 \le i \le n$, α_i is an $\mathbb{F}$-automorphism of the ring $R_{i-1} = \mathbb{F}[x_1;\alpha_1,\delta_1][x_2;\alpha_2,\delta_2]\cdots[x_{i-1};\alpha_{i-1},\delta_{i-1}]$ and δ_i is an $\alpha_i-\mathbb{F}-$derivation of R_{i-1}. Then R is the $\mathbb{F}$-algebra generated by $x_1, x_2, \ldots, x_n$ subject to the $\frac{1}{2}n(n-1)$ relations*

$$x_i x_j = \alpha(x_j)x_i + \delta_i(x_j), \quad 1 \le j < i \le n.$$

Proof. See [6, Proposition 1]. □

Definition 1.3.12. [8, p. 16] Let $\mathbb{F}$ be a field and $q \in \mathbb{F}^\times$. The **quantized coordinate ring** of $(\mathbb{F}^\times)^2$, or **quantum torus**, is the $\mathbb{F}$-algebra $\mathcal{O}_q((\mathbb{F}^\times)^2)$ presented by generators $x,\ x',\ y,\ y'$ and relations

$$xx' = x'x = yy' = y'y = 1, \quad xy = qyx.$$

In brief, we may say that $\mathcal{O}_q((\mathbb{F}^\times)^2)$ is presented by generators $x^{\pm 1}$ and $y^{\pm 1}$ satisfying $xy = qyx$. The quantum torus $\mathcal{O}_q((\mathbb{F}^\times)^2)$ is a skew Laurent polynomial ring $\mathbb{F}[y^{\pm 1}][x^{\pm 1};\alpha]$, with $\alpha(y) = qy$.

Definition 1.3.13. [8, p. 41] Let $q \in \mathbb{F}^{\times}$ (with $\mathbb{F}$ an arbitrary field) be any non-zero scalar such that $q \neq \pm 1$. The **quantized enveloping algebra** $U_q(sl_2(\mathbb{F}))$ is the $\mathbb{F}-$algebra presented by four generators $E,\ F,\ K,\ K^{-1}$ and five relations

$$KK^{-1} = K^{-1}K = 1, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$KE = q^2 EK, \qquad KF = q^{-2} FK.$$

Theorem 1.3.14. *Let R be a ring and $T = R[x^{\pm 1}; \sigma]$ where $\sigma \in \operatorname{Aut} R$. Then T is a simple ring if and only if R has no proper non-zero $\sigma-$stable ideals and no non-zero power of σ is an inner automorphism of R.*

Proof. See [19, Theorem 1.8.5]. □

Example 1.3.15. [19, 1.8.7] Let α be the $\mathbb{F}-$automorphism of $\mathbb{F}[y]$ such that $\alpha(y) = y + 1$ and let $R = \mathbb{F}[y][x^{\pm 1}; \alpha]$. Suppose that $\operatorname{char} \mathbb{F} = 0$. The ring R is generated by $x,\ x^{-1}$ and y subject to the relation $xy - yx = x$. As $\mathbb{F}[y]$ is commutative and $\alpha^n(y) = y + n$, no non-zero power of α is an inner automorphism. It is easy to check that $\mathbb{F}[y]$ has no proper non-zero $\alpha-$stable ideals. Then $R[x^{\pm 1}; \alpha]$ is simple by Theorem 1.3.14.

Definition 1.3.16. Let R be a finitely generated $\mathbb{F}$-algebra and let $x_1, x_2, \ldots, x_n$ be generators of R. We say that R has a **PBW-basis** with respect to $x_1, x_2, \ldots, x_n$ if the standard monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, each $i_j \geq 0$, form a basis for R over $\mathbb{F}$. Here PBW stands for Poincare-Birkhoff-Witt whose theorem [19] says that the enveloping algebra of a finite dimensional Lie algebra has a PBW-basis.

Theorem 1.3.17. *If $R = \mathbb{F}[x_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$ is an iterated skew polynomial ring, where each α_i is an $\mathbb{F}$-automorphism of $\mathbb{F}[x_1][x_2; \alpha_2, \delta_2] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}]$ and each δ_i is an α_i-derivation of R, then R has a PBW-basis with respect to $x_1, x_2, \ldots, x_n$.*

Proof. Write $R_i = \mathbb{F}[x_1][x_2; \alpha_2, \delta_2] \cdots [x_i; \alpha_i, \delta_i]$ and $R_0 = \mathbb{F}$. For $j \geq 1$, each element of R_i has a unique expression $\sum_{j=0}^{n} f_j x_i^j$, $f_j \in R_{i-1}$. The result follows by induction. □

1.4 Normal elements and prime ideals

Definition 1.4.1. A **normal element** in a ring R is any element $x \in R$ such that $xR = Rx$. In particular, of course, any central element is normal.

Definition 1.4.2. A **prime ideal** in a ring R is any proper ideal P of R such that whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A **prime ring** is a ring in which 0 is a prime ideal.

Definition 1.4.3. [19, 0.1.2] A ring R is called an **integral domain** if the product of non-zero elements is always non-zero; and an ideal A of any ring R is **completely prime** if R/A is an integral domain.

Definition 1.4.4. [17, p. 34] Let P be a prime ideal of the ring R. The **height** of P is

$$ht(P) = \sup\{m : \text{there is a chain of prime ideals } P \supsetneqq P_1 \supsetneqq \dots \supsetneqq P_m\}.$$

The following result is useful in showing that certain homomorphic images of skew polynomial rings are integral domains.

Proposition 1.4.5. *Let α be an automorphism and δ be an α-derivation of a domain A. Let $R = A[x; \alpha, \delta]$. Let c be a central element of R of the form $dx + e$, where $d, e \in A$ and $d \neq 0$. Then $ad = d\alpha(a)$ for all $a \in A$ so is a normal element. Furthermore, if e is a non-zero divisor modulo dA then R/cR is a domain.*

Proof. See [11, Proposition 1]. □

Theorem 1.4.6 (Hilbert's Nullstellensatz Theorem). *Let $R = \mathbb{F}[x_1, x_2, \dots, x_n]$ be the polynomial ring over $\mathbb{F}$ in the $n(> 0)$ indeterminates $x_1, x_2, \dots, x_n$. The ideal M is a maximal ideal if and only if there exist $a_1, a_2, \dots, a_n$ such that $M = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$.*

Proof. See [21, Theorem 14.6]. □

Theorem 1.4.7. *In a right or left Noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals that equals zero.*

Proof. See [8, Theorem 3.4]. ☐

Corollary 1.4.8. *For a ring R, the following conditions are equivalent:*

(i) R is prime and left Artinian.

(ii) R is simple and left Artinian.

Proof. See [8, Corollary 4.18]. ☐

1.5 Gelfand-Kirillov Dimension

Let A be an affine $\mathbb{F}$-algebra and let V be a finite dimensional generating subspace for A containing $1 = 1_A$. Then there is an ascending chain of subspaces

$$\mathbb{F} = V^0 \subseteq V \subseteq V^2 \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} V^n = A.$$

Suppose that V is spanned by $v_1, \ldots, v_m$. Then V^n is the subspace spanned by all monomials in $v_1, v_2, \ldots, v_m$ of length not greater than n.

Definition 1.5.1. [17, p. 14] The **Gelfand-Kirillov dimension** of an affine $\mathbb{F}$-algebra A, denoted by $GK\dim(A)$, is defined as

$$GK\dim(A) = \lim_{n\to\infty} \sup \log_n(\dim(V^n)),$$

where V is a finite-dimensional generating subspace for A. This is independent of choice of V, see [17].

Example 1.5.2. [19, 8.1.15] If $R = \mathbb{F}[x_1, x_2, \ldots, x_n]$, the commutative polynomial ring, then $GK\dim(R) = n$.

Proposition 1.5.3. *Let R be any $\mathbb{F}$-algebra. If $R \subseteq R'$ with R' an $\mathbb{F}$-algebra finitely generated as a right, or left, R-module then $GK\dim(R') = GK\dim(R)$.*

Proof. See [17, Proposition 5.5]. ☐

Corollary 1.5.4. *Let R be a left Noetherian $\mathbb{F}-$ algebra and $\gamma \in \mathrm{Aut}_{\mathbb{F}} R$ be of finite order n. Suppose that $n^{-1} \in \mathbb{F}$. Then $GK\dim(R^{\gamma}) = GK\dim(R)$.*

Proof. It follows from Proposition 1.2.3 and Proposition 1.5.3. ☐

Proposition 1.5.5. *Let R be an algebra and Ω be a multiplicatively closed set of regular elements that is contained in the centre of R. Then*

$$GK\dim(R\Omega^{-1}) = GK\dim(R).$$

Proof. See [17, Proposition 4.2]. ☐

Corollary 1.5.6. *For any $\mathbb{F}$-algebra R,*

$$GK\dim(R[x, x^{-1}]) = GK\dim(R) + 1.$$

Proof. See [19, Corollary 8.2.15]. ☐

Theorem 1.5.7. *Let A be a finitely generated $\mathbb{F}$-algebra, let α be an automorphism of A and let δ be an α-derivation of A. Suppose that there is a finite dimensional generating subspace B for A containing 1 such that $\delta(B) \subseteq B^2$ and $\alpha(B) \subseteq B$. Set $R = A[x; \alpha, \delta]$ and $S = A[x^{\pm 1}; \alpha]$. Then*

$$GK\dim(R) = GK\dim(S) = GK\dim(A) + 1.$$

Proof. See [1, Theorem 3.1.3]. ☐

Proposition 1.5.8. *Let A be an $\mathbb{F}$-algebra with $\mathbb{F}$-derivation δ such that each finite dimensional subspace of A is contained in a δ-stable finitely generated subalgebra of A. Then*

$$GK\dim(A[x; \delta]) = GK\dim(A) + 1.$$

Proof. See [17, Proposition 3.5] □

Theorem 1.5.9. *Let A be a Noetherian $\mathbb{F}$-algebra. If P is a prime ideal of A, then*

$$GK\dim(A) \geq GK\dim(A/P) + ht(P).$$

Proof. See [17, Corollary 3.16]. □

Example 1.5.10. Consider the quantum torus and let V be the finite-dimensional generating subspace $\mathrm{Sp}(\{1, x, y, x^{-1}, y^{-1}\})$. Then $V^n = \mathrm{Sp}(\{x^i y^j : -n \leq i + j \leq n\})$. This is independent of $q \in \mathbb{F}\backslash\{0\}$ and therefore $GK\dim(\mathcal{O}_q((\mathbb{F}^\times)^2)) = GK\dim(\mathbb{F}[x^{\pm 1}, y^{\pm 1}]) = 2$ by two applications of Corollary 1.5.6.

1.6 Diamond Lemma

We shall use the Diamond Lemma of Bergman [2] to find bases for some of the $\mathbb{F}$-algebras occurring in the thesis. Here we list the basic definitions and ideas. This is based on [3, I.11].

Let A be a finitely generated $\mathbb{F}$-algebra with generators $x_1, x_2, \ldots, x_n$. Let F be the free algebra $\mathbb{F}\langle x_1, x_2, \ldots, x_n\rangle$ and let W be the free monoid on $\mathbb{F}\{x_1, x_2, \ldots, x_n\}$. Thus W consists of words $x_{i_1} x_{i_2} \ldots x_{i_m}$ in the x_i's and it is a basis for F. Suppose that A is presented by relations p_σ, $\sigma \in \Sigma$, for some set Σ. Each p_σ can be written in the form $w_\sigma - f_\sigma$ where $w_\sigma \in W$ and $f_\sigma \in F$. In this situation, we have the following definitions.

- The subset $S = \{(w_\sigma, f_\sigma) : \sigma \in \Sigma\}$ of $W \times F$ is a **reduction system**.
- For $\sigma \in \Sigma$ and $a, b \in W$, the $\mathbb{F}$-linear map sending $aw_\sigma b$ to $af_\sigma b$ and fixing all other words in W is called an **elementary reduction**.
- Any composition of finitely many elementary reductions is called a **reduction**.
- An element $f \in F$ is **irreducible** (with respect to S) if $r(f) = f$ for all reductions, in other words when f is expressed as a linear combination of

words in W, none of the words w_σ is a subword of any of the words that occur in f.

We need to order the elements of W.

(i) A **semigroup ordering** on W is a partial order $\leq$ such that

$$b < b' \Rightarrow abc < ab'c$$

for all $a,\ b,\ b',\ c \in W$.

(ii) A semigroup ordering $\leq$ on W is **compatible** with S provided that for all $\sigma \in \Sigma$, the element f_σ is a linear combination of words $w < w_\sigma$.

(iii) A semigroup ordering on W satisfies the **descending chain condition (DCC)** if for every chain $b_1 \geq b_2 \geq b_3 \ldots$ of elements in W, there is an integer m such that $b_j = b_m$ for all $j \geq m$. Equivalently, every non-empty subset of W has a minimal element under $\leq$.

(iv) The **lexicographic ordering** $\leq_{\text{lex}}$ on W is the lexicographic order on sequences of positive integers transferred to monomials in W via indices. That is,

$$a = x_{i(1)}x_{i(2)}\ldots x_{i(s)} \leq_{\text{lex}} b = x_{j(1)}x_{j(2)}\ldots x_{j(t)}$$

if and only if either

- $s \leq t$ and $i(l) = j(l)$ for all $l \leq s$, or
- there is some $u \leq \min\{s,t\}$ such that $i(l) = j(l)$ for $l < u$ while $i(u) < j(u)$.

(v) The **length lexicographic ordering** on W is the following modification of the lexicographic ordering: $a \leq b$ if and only if either

- $\text{length}(a) < \text{length}(b)$, or
- $\text{length}(a) = \text{length}(b)$ and $a \leq_{\text{lex}} b$.

The semigroup ordering $\leq$ on W satisfies the DCC whereas $\leq_{\text{lex}}$ does not if $n > 1$. For example,

$$x_2 > x_1x_2 > x_1^2x_2 > x_1^3x_2 > \cdots .$$

Example 1.6.1. In the $\mathbb{F}$-algebra A generated by x, y and z subject to the relations

$$\begin{aligned} xy - qyx &= z \\ yz - qzy &= x \\ zx - qxz &= y \end{aligned}$$

where $q \in \mathbb{F}\backslash\{0\}$. Then the length lexicographic ordering is compatible with S when we order the indeterminates $x > y > z$. Here $\sigma_1 := (xy, qyx+z)$, $\sigma_2 := (yz, qzy+x)$ and $\sigma_3 := (xz, q^{-1}(zx - y))$ form a reduction system S for which the irreducible elements of F are linear combinations of the monomials that do not contain xy, yz or xz as a subword, that is of the standard monomials $z^iy^jx^k$.

The application of the Diamond Lemma involves handling the so-called ambiguities which are defined as follows;

- An **overlap ambiguity** is a 5-tuple (a, b, c, σ, τ) in $W^3 \times \Sigma^2$ such that $ab = w_\sigma$ and $bc = w_\tau$. The ambiguity lies in the fact that abc has two immediate reductions:
$$r_{1,\sigma,c}(abc) = f_\sigma c \quad \text{and} \quad r_{a,\tau,1}(abc) = af_\tau.$$
- An **inclusion ambiguity** is a 5-tuple (a, b, c, σ, τ) in $W^3 \times \Sigma^2$ such that $abc = w_\sigma$ and $b = w_\tau$. In this case,
$$r_{1,\sigma,c}(abc) = f_\sigma \quad \text{and} \quad r_{a,\tau,1}(abc) = af_\tau c.$$
- An overlap (respectively, inclusion) ambiguity (a, b, c, σ, τ) is **resolvable** if and only if there exist reductions r, r' such that $r(f_\sigma c) = r'(af_\tau)$ (respectively, $r(f_\sigma) = r'(af_\tau c)$).

In Example 1.6.1, $(x, y, z, \sigma_1, \sigma_2)$, or simply xyz, is an overlap ambiguity and is the only ambiquity. We see that

$$\begin{aligned}(xy)z &= (qyx + z)z = qy(q^{-1}zx - q^{-1}y) + z^2 \\ &= qzyx + x^2 - y^2 + z^2\end{aligned}$$

and

$$\begin{aligned}x(yz) &= x(qzy + x) = q(q^{-1}zx - q^{-1}y)y + x^2 \\ &= qzyx + x^2 - y^2 + z^2\end{aligned}$$

So this ambiquity is resolvable.

We can now state the Diamond Lemma.

Lemma 1.6.2 (Diamond Lemma)**.** *[3, I.11] Let $F = k\langle X\rangle$ be a free algebra on a set X and W the free monoid on X. Let $S = \{(w_\sigma, f_\sigma) : \sigma \in \Sigma\}$ be a reduction system for F, and $\leq$ a semigroup ordering on W which is compatible with S and satisfies the DCC. Assume that all overlap and inclusion ambiguities are resolvable. Then the cosets $\overline{w}$, for irreducible words $w \in W$, form a basis for the factor algebra $F/\langle w_\sigma - f_\sigma : \sigma \in \Sigma\rangle$.*

Proof. See [2, Theorem 1.2]. □

In Example 1.6.1, the Diamond Lemma shows that A has a PBW-basis $\{z^i y^j x^k\}$.

1.7 Associated graded rings

The method of filtered rings and associated graded rings will be very useful for us. These methods are outlined below; our main source is [19].

Definition 1.7.1. [19, 1.6.1] A **filtered ring** is a ring R with a family $\{R_i\}_{i\geq 0}$ of additive subgroups of R such that

(i) $1 \in R_0$;

(ii) $R_i \subseteq R_j$ for all $i \leq j$;

(iii) $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$;

(iv) $R = \bigcup_{i \geq 0} R_i$.

The family $\{F_i\}$ is called a **filtration** of R.

Definition 1.7.2. [19, 1.6.3] A **graded ring**, or $\mathbb{N}-$**graded ring**, is a ring T together with a family $\{T_n\}_{n \geq 0}$ of additive subgroups of T such that

(i) $T_i T_j \subseteq T_{i+j}$, for all $i, j \geq 0$ and

(ii) $T = \bigoplus_n T_n$, as an abelian group.

The family $\{T_n\}$ is called a **grading** of T and a non-zero element of T_n is said to be **homogeneous** of degree n.

Definition 1.7.3. Let R be a filtered ring with filtration $\{R_i\}_{i \geq 0}$. The **associated graded ring** is

$$\operatorname{gr} R = \bigoplus_{i \geq 0} R_i / R_{i-1}, \quad \text{(as additive groups)}, \quad \text{where,} \quad R_{-1} = 0,$$

equipped with obvious addition and with multiplication given by

$$(r + R_{i-1})(s + R_{j-1}) = rs + R_{i+j-1} \quad \text{for } r \in R_i,\ s \in R_j.$$

When $r \in R_i / R_{i-1}$, we write $\overline{r} = r + R_{i-1} \in \operatorname{gr} R$.

Proposition 1.7.4. *[19, 1.6.6] Let S be a filtered ring and grS be the associated graded ring of S.*

(i) If grS is an integral domain then S is an integral domain.

(ii) If grS is right (left) Noetherian, then S is right (left) Noetherian.

Proof. See [19, Proposition 1.6.6 and Proposition 1.6.9]. □

In this thesis, the filtrations considered occur in the following way, which is slightly more general than that described in [3, I.12.2(c)]. Let A be a finitely generated $\mathbb{F}$-algebra with generators $x_1, x_2, \ldots, x_n$. Let W be the free monoid on $\{x_1, x_2, \ldots, x_n\}$ and let F be the free algebra $\mathbb{F}\langle x_1, x_2, \ldots, x_n\rangle$. By a **degree function** on W, we mean a monoid homomorphism $d : W \to (\mathbb{N}^+, +)$. Such a function is determined by its values on $x_1, x_2, \ldots, x_n$. Set $A_0 = \mathbb{F}$, and, for $i \geq 1$, let A_i be the $\mathbb{F}$-subspace of A spanned by the words $m \in W$ with $d(m) \leq i$. Then $A_0 \subseteq A_1 \subseteq A_2 \ldots$ is a filtration of A. We shall call this the **standard filtration** with respect to the generators $x_1, x_2, \ldots, x_n$ and the degree function d.

Suppose that A has a presentation F/I where I is the ideal generated by the elements $w_\sigma - f_\sigma$, for some reduction system $S = \{(w_\sigma, f_\sigma)\}$. Let $\leq$ be a semigroup ordering on W that has DCC and is compatible with S. We say that a degree function $d : W \to \mathbb{N}^+$ is **compatible** with S if, for each $(w_\sigma, f_\sigma) \in S$, f_σ is a linear combination of words m with $d(m) \leq d(w_\sigma)$. If $\leq$ is the length lexicographic ordering and $d(x_i) = 1$ for each i then compatibility of d with S is a consequence of compatibility of $\leq$ with S.

Set $B_0 = \mathbb{F}$, and, for $i \geq 1$, let B_i be the $\mathbb{F}$-subspace spanned by the irreducible words m with $d(m) \leq i$. Thus $B_i \subseteq A_i$. We claim that $B_i = A_i$ for each i. Suppose not. By DCC, there exists a word $m \in W$ that is minimal, under $\leq$, with the property that $m \in A_{d(m)} \backslash B_{d(m)}$. Then m cannot be irreducible so there exists $(w_\sigma, f_\sigma) \in S$ such that $m = aw_\sigma b$ for some $a, b \in W$. Then $m = af_\sigma b$ is a linear combination of words $w < m$, with $d(w) \leq d(m)$, whence $m \in B_{d(m)}$, contradicting the minimality of m. Therefore $A_i = B_i$ for all i.

Suppose now that all ambiguities in S are resolvable, so that by the Diamond Lemma, the irreducible words form a basis for A. Denote by $\mathcal{M}$ the set of irreducible words. In the filtration $\{B_i\}$, each B_i has basis $B_i \cap \mathcal{M}$. Hence, in $\operatorname{gr} A$, each summand $A_i/A_{i-1} = B_i/B_{i-1}$ has a basis consisting of the elements $\bar{m}$ where $m \in (B_i \cap \mathcal{M}) \backslash B_{i-1}$ and so $\operatorname{gr} A$ has a basis consisting of the elements $\bar{m}$ where $m \in \mathcal{M}$. Therefore there is a vector space isomorphism $\psi : A \to \operatorname{gr} A$ given by $\psi(m) = \bar{m}$ for all $m \in \mathcal{M}$.

Example 1.7.5. Let A be the $\mathbb{F}$-algebra from Example 1.6.1. We have seen that there is a PBW-basis $\{z^i y^j x^k\}$. Take the degree function d with $d(x) = d(y) = d(z) = 1$. The associated graded ring, $\operatorname{gr} A$, is generated by $\bar{x}, \bar{y}$ and $\bar{z}$ and these satisfy the relations

$$\begin{aligned} \bar{x}\bar{y} - q\bar{y}\bar{x} &= 0 \\ \bar{y}\bar{z} - q\bar{z}\bar{y} &= 0 \\ \bar{z}\bar{x} - q\bar{x}\bar{z} &= 0. \end{aligned}$$

Let B be the $\mathbb{F}-$algebra generated by X, Y and Z subject to these relations. There is a surjection $\phi : B \to \operatorname{gr} A$ with $\phi(X) = \bar{x}, \phi(Y) = \bar{y}$ and $\phi(Z) = \bar{z}$. Using the vector space isomorphism $\psi : A \to \operatorname{gr} A$, we see that the standard monomials $Z^i Y^j X^k$ form a basis for $\operatorname{gr} A$. Also B has a PBW-basis, by Theorem 1.3.17, so ϕ is an isomorphism and $\operatorname{gr} A \simeq B$.

Given a degree function d we can modify the length lexicographic ordering by ordering words first by the degree function d and then lexicographically with a specified order $x_1 < x_2 < \ldots < x_n$ for the generators. Thus $m \leq m'$ if and only if either $d(m) < d(m')$ or $d(m) = d(m')$ and $m \leq_{\text{lex}} m'$. We shall refer to this as the **d-length lexicographic ordering** with respect to $x_1 < x_2 < \ldots < x_n$. It is clearly a semigroup ordering and if $d(x_i) > 0$ for each i it has DCC. It may have DCC when some $d(x_i) = 0$ as the following lemma shows.

Lemma 1.7.6. *Let M be the free monoid on three generators x, y, z and let d be the degree function such that $d(x) = 0$ but $d(y) = d(z) = 1$. The d-lexicographic ordering on M, with $x > y > z$ and $mx^i < mx^j$ when $m \in M$ and $i < j$, has DCC.*

Proof. It is clear that $\leq$ is a semigroup ordering. We claim that $\leq$ satisfies DCC. It suffices to show that if C is a non-empty subset of M then C has a minimal element. Let $p(C) = \min(\{n \in \mathbb{N}_0 : d(c) = n \quad \text{for some} \quad c \in C\})$ and let $C_{p(C)} = \{c \in C : d(c) = p(C)\}$. We proceed by induction on $p(C)$. When $p(C) = 0$, as the words in C_0 are powers of x and $1 < x < x^2 < x^3 \ldots$, C has a minimal element x^j where $j = \min(\{i : x^i \in C\})$. Now let $k > 0$ and suppose that if D is a non-empty subset of M with $p(D) = k-1$, D has a minimal element. Let C be a non-empty subset of M such that $p(C) = k$. Each word in C_k must begin either $x^i z$ or $x^i y$ for some $i \geq 0$. Choose the minimal i occurring in this way and let E, respectively F, denote the set of words in C_k beginning $x^i z$, respectively $x^i z$. Suppose that $E \neq \emptyset$ and let B be the set of all words b such that $x^i yb \in C_k$. As $d(z) = 1$, it follows that $d(b) = k-1$ for all $b \in B$ so B has a minimal element, b say. Then $x^i zb$ is minimal in E and in C. If $E = \emptyset$ then, repeating the argument with F replacing E, C has a minimal element $x^i yb$ where b is minimal in the set of words a such that $x^i ya \in F$. Thus $\leq$ has DCC. □

Definition 1.7.7. A $\mathbb{Z}_2-$**graded ring** is a ring $T = T_0 \oplus T_1$, where T_0 is a subring of T, T_1 is a right and left T_0-module (so $T_0T_1 \subseteq T_1$ and $T_1T_0 \subseteq T_1$) and $T_1T_1 \subseteq T_0$. The subring T_0 is called the **even part** of T and T_1 is said to be the **odd part** of T.

Definition 1.7.8. A left $T-$module M over a $\mathbb{Z}_2-$graded ring T is $\mathbb{Z}_2-$**graded** if $M = M_0 \oplus M_1$ where M_0 and M_1 are T_0-submodules (so $T_0M_0 \subseteq M_0$ and $T_0M_1 \subseteq M_1$) such that

$$T_1M_0 \subseteq M_1 \quad \text{and} \quad T_1M_1 \subseteq M_0.$$

The submodule M_0 is called the **even part** of M and M_1 is said to be the **odd part** of M.

1.8 Leonard Triples

In this section, we shall give the definition of Leonard triples, introduced by Terwilliger [22].

Definition 1.8.1. An $n \times n$ matrix $A = (a_{ij})$ is called **tridiagonal** whenever each non-zero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

In other words, $a_{ij} = 0$ whenever $|i-j| > 1$. A tridiagonal matrix is said to be **irreducible** whenever all entries immediately above and below the main diagonal are non-zero. In other words, $a_{i(i+1)} \neq 0$ and $a_{(i+1)i} \neq 0$ for $1 \leq i < n$.

Definition 1.8.2. Let V be a finite-dimensional vector space over $\mathbb{F}$ and let $A : V \to V$ denote a linear transformation. Let θ be an eigenvalue of A. Let $W = \{v \in V : Av = \theta v\}$. Then W is called the **eigenspace** of A (for the eigenvalue θ). Let $U = \{v \in V : (A - \theta\,\mathrm{id})^n v = 0 \text{ for some } n \geq 1\}$. The subspace U is called the **generalized eigenspace** of A for θ. We say A is **diagonalisable** whenever V is spanned by the eigenspaces of A.

Definition 1.8.3. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. A **Leonard triple** on V means a three-tuple of linear transformations $x : V \to V$, $y : V \to V$ and $z : V \to V$ that satisfy the conditions (i)-(iii) below.

(i) There exists a basis for V with respect to which the matrix representing x is diagonal and the matrices representing y and z are each irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing y is diagonal and the matrices representing x and z are each irreducible tridiagonal.

(iii) There exists a basis for V with respect to which the matrix representing z is diagonal and the matrices representing x and y are each irreducible tridiagonal.

1.9 Poisson algebras and Poisson modules

Let A be a commutative algebra over $\mathbb{C}$.

Definition 1.9.1. A **Poisson bracket** on A is a Lie algebra bracket $\{-,-\}$ satisfying the Leibniz rule

$$\{ab, c\} = a\{b, c\} + \{a, c\}b \quad \text{for all } a, b, c \in A.$$

The pair $(A, \{-,-\})$ is called a **Poisson algebra**.

Thus the axioms for a Poisson algebra are the axioms for a commutative associative algebra together with the following:

$$\begin{aligned} \{a,b\}+\{b,a\} &= 0 \\ \{\{a,b\},c\}+\{\{c,a\},b\}+\{\{b,c\},a\} &= 0 \\ \{ab,c\} &= a\{b,c\}+\{a,c\}b \end{aligned}$$

for all $a,\ b,\ c \in A$.

Definition 1.9.2. Let $a \in A$. Then $\{a,-\} : A \to A$ is a derivation of A. We denote it $\mathrm{ham}(a)$ and call it the **hamiltonian** vector field for a.

Definition 1.9.3. Let I be an ideal of a Poisson algebra A. Then I is a **Poisson ideal** if $\{A, I\} \subseteq I$, that is, $\{r,x\} \in I$ for all $x \in I,\ r \in A$.

Note that I is a Poisson ideal of A it is easy to check that A/I is a Poisson algebra under the obvious bracket $\{a+I, b+I\} = \{a,b\}+I$.

Definition 1.9.4. Let P be an ideal of a Poisson algebra A. Then P is a **prime Poisson ideal** if P is a prime ideal and a Poisson ideal of A.

Definition 1.9.5. Let P be an ideal of a Poisson algebra A. Then P is a **Poisson-prime ideal** if

(i) P is a Poisson ideal;

(ii) for all Poisson ideals $I,\ J \subseteq A$,

$$IJ \subseteq P \quad \text{implies that } I \subseteq P \text{ or } J \subseteq P.$$

Fortunately, for finitely generated algebras the notions of prime Poisson and Poisson-prime are equivalent. This is well-known, for example see [4], but we give a proof below.

Theorem 1.9.6. *Let P be an ideal of a finitely generated Poisson algebra A. Then the following are equivalent:*

(i) P is a prime Poisson ideal.

(ii) P is a Poisson-prime ideal,

Proof. (i)$\Rightarrow$(ii). Let P be a prime Poisson ideal of A. Let I and J be Poisson ideals of A such that $IJ \subseteq P$. Since P is a prime ideal, it follows that $I \subseteq P$ or $J \subseteq P$. Hence P is a Poisson-prime ideal.

(ii)$\Rightarrow$(i). Let P be a Poisson-prime ideal of A. We want to show that P is a prime ideal of A. Note that A is a commutative Noetherian ring, so by Theorem 1.4.7, there exist only finitely many primes $Q_1, Q_2, \ldots, Q_n$ that are minimal over P. There is a finite product of the Q_i's contained in P. If we can show that each Q_i is a Poisson ideal then $Q_i \subseteq P \subseteq Q_i$ for some i, and $P = Q_i$. Hence it suffices to prove that each Q_i is a Poisson ideal of A. Let $a \in A$ and let $\delta = \operatorname{ham} a$ be the hamiltonian vector field for a. Let $Q = \{r \in A : \delta^j(r) \in Q_i \text{ for all } j \geq 0\}$. Taking $j = 0$ and $r \in Q$, then $r = \delta^0(r) \in Q_i$, so $Q \subseteq Q_i$. Since P is a Poisson ideal, $P \subseteq Q \subseteq Q_i$. Let $b, c \in A$ be such that $b \notin Q$ and $c \notin Q$, so we have $\delta^j(b) \notin Q_i$ and $\delta^k(c) \notin Q_i$ for some minimal $j,\ k$. Observe that

$$\delta^{j+k}(bc) = b\delta^{j+k}(c) + \cdots + \binom{j+k}{j} \delta^j(b)\delta^k(c) + \cdots + \delta^{j+k}(b)c.$$

Since $\delta^j(b) \notin Q_i$ and $\delta^k(c) \notin Q_i$, $\delta^j(b)\delta^k(c) \notin Q_i$. Since $b, \delta(b), \ldots, \delta^{j-1}(b) \in Q_i$, and $\delta^{k-1}(c), \ldots, \delta(c), c \in Q_i$, it follows that $\delta^{j+k}(bc) \notin Q_i$. Hence $bc \notin Q$, and Q is a prime ideal. Since Q_i is minimal over P, this shows that $Q = Q_i$. By definition, $\delta(Q) \subseteq Q$ so $\delta(Q_i) \subseteq Q_i$, that is, $\{a, Q_i\} \subseteq Q_i$. Thus Q_i is Poisson for $1 \leq i \leq n$. Therefore, as indicated above, $P = Q_i$ for some i and therefore P is prime. □

Definition 1.9.7. A Poisson algebra A is said to be **simple** if it has no other Poisson ideal than (0) and A.

Note that there is more than one definition of Poisson module in the literature. Here is the one introduced by D. R. Farkas [7].

Definition 1.9.8. Let A be a commutative Poisson algebra with Poisson bracket $\{-,-\}$. We shall say that an A-module M is a **Poisson module** if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that

(i) $\{a, a'm\}_M = \{a, a'\}m + a'\{a, m\}_M$ for all $a, a' \in A$ and all $m \in M$.

(ii) $\{aa', m\}_M = a\{a', m\}_M + a'\{a, m\}_M$ for all $a, a' \in A$ and all $m \in M$.

(iii) $\{\{a, a'\}, m\}_M = \{a, \{a', m\}_M\}_M - \{a', \{a, m\}_M\}_M$ for all $a, a' \in A$ and all $m \in M$.

Remark 1.9.9. Of these, [4] only requires (i). Given (i), condition (ii) is equivalent to

(ii)$'$: $\{aa', m\}_M = \{a, a'm\}_M + \{a', am\}_M$ for all $a, a' \in A$ and all $m \in M$.

Proof. Suppose that (i) and (ii) hold. Let $a, a' \in A$ and $m \in M$. By (i),

$$\begin{aligned} \{a, a'm\}_M &= \{a, a'\}m + a'\{a, m\}_M && (1.1)\\ \{a', am\}_M &= \{a', a\}m + a\{a', m\}_M && (1.2) \end{aligned}$$

By adding (1.1) and (1.2), we have

$$\begin{aligned} \{a, a'm\}_M + \{a', am\}_M &= \{a, a'\}m + a'\{a, m\}_M + \{a', a\}m + a\{a', m\}_M\\ &= \{a, a'\}m + a'\{a, m\}_M - \{a, a'\}m + a\{a', m\}_M\\ &= a'\{a, m\}_M + a\{a', m\}_M = \{aa', m\}_M \quad \text{by } (ii). \end{aligned}$$

Conversely, suppose that (i) and (ii)$'$ hold and rewrite (1.1) and (1.2) as

$$\begin{aligned} a'\{a, m\}_M &= \{a, a'm\}_M - \{a, a'\}m \quad \text{and} && (1.3)\\ a\{a', m\}_M &= \{a', am\}_M - \{a', a\}m = \{a', am\}_M + \{a, a'\}m. && (1.4) \end{aligned}$$

Again, by adding (1.3) and (1.4), we have

$$a'\{a, m\}_M + a\{a', m\}_M = \{a, a'm\}_M + \{a', am\}_M.$$

Therefore, by (ii)$'$, $\{aa', m\}_M = a\{a', m\}_M + a'\{a, m\}_M$ for all $a, a' \in A$ and $m \in M$. □

Definition 1.9.10. A submodule N of a Poisson module M is called a **Poisson submodule** if for all $a \in A$, $n \in N$, $\{a, n\}_M \in N$.

Definition 1.9.11. A Poisson module M is said to be a **simple Poisson module** if $M \neq 0$ and it has no other Poisson submodule than (0) and M.

We next describe a way in which a noncommutative algebra can induce a Poisson structure on a commutative algebra.

Definition 1.9.12 (Quantization)**.** [3, III.5.4] Let R be a commutative algebra and let $h \in R$. Let A be an $R-$algebra and suppose that h is not a zero divisor in A, and that $\overline{A} := A/hA$ is a commutative $\mathbb{F}$-algebra. For $\alpha, \beta \in \overline{A}$, there exist $a, b \in A$ such that $\alpha = \overline{a} = a + hA$ and $\beta = \overline{b} = b + hA$. By commutativity of $\overline{A}$,

$$[a, b] = h\gamma(a, b)$$

for a unique element $\gamma(a, b)$ of A. Here $[a, b]$ means $ab - ba$. Define

$$\{\alpha, \beta\} = \overline{\gamma(a, b)}.$$

This does not depend on the choice of a and b and it defines a Poisson bracket on $\overline{A}$. We say that A is a **quantization** of the Poisson algebra $\overline{A}$.

Chapter 2

Rings of Invariants

In this chapter, we shall determine the ring of invariants for the automorphisms of the localized enveloping algebra and the quantum torus discussed in the Introduction. When q is replaced by 1 in the quantum torus, our result agree with the result in Example 1.2.4.

2.1 Notation

Definition 2.1.1. Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$. We say that α is **γ-reversible** if $\gamma\alpha\gamma^{-1} = \alpha^{-1}$.

Proposition 2.1.2. *Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$. The following are equivalent:*

(i) α *is* γ*-reversible;*

(ii) $(\alpha\gamma)^2 = \mathrm{id}_R$*;*

(iii) $(\gamma\alpha)^2 = \mathrm{id}_R$*;*

(iv) $\alpha = \gamma\tau$ *for some* $\tau \in \mathrm{Aut}(R)$ *such that* $\tau^2 = \mathrm{id}_R$*;*

(v) $\alpha = \tau'\gamma$ *for some* $\tau' \in \mathrm{Aut}(R)$ *such that* $\tau'^2 = \mathrm{id}_R$*.*

Proof. (i)$\Rightarrow$ (ii). Suppose that (i) holds. Then $(\alpha\gamma)^2 = \alpha\gamma\alpha\gamma = \alpha\alpha^{-1} = \mathrm{id}_R$.

(ii)$\Rightarrow$(iii). If (ii) holds, then $(\gamma\alpha)^2 = \gamma\alpha\gamma\alpha\gamma^2 = \gamma(\alpha\gamma)^2\gamma = \gamma^2 = \mathrm{id}_R$.

(iii)$\Rightarrow$(i). Suppose that $(\gamma\alpha)^2 = \mathrm{id}_R$. Then $\gamma\alpha\gamma = \gamma\alpha\gamma\alpha\alpha^{-1} = (\gamma\alpha)^2\alpha^{-1} = \alpha^{-1}$.

(iii)$\Rightarrow$(iv). Let $\tau = \gamma'\alpha \in \mathrm{Aut}(R)$. Thus $\alpha = \gamma\tau$. Observe that if (iii) holds then $\mathrm{id}_R = (\gamma\alpha)(\gamma\alpha) = \gamma(\gamma\tau)\gamma(\gamma\tau) = \tau^2$.

(iv)$\Rightarrow$(ii). If $\tau^2 = \mathrm{id}_R$, then $(\alpha\gamma)(\alpha\gamma) = (\gamma\tau)\gamma(\gamma\tau)\gamma = \gamma\tau^2\gamma = \mathrm{id}_R$.

(ii)$\Rightarrow$(v). By assumption, choose $\tau' \in \mathrm{Aut}(R)$ such that $\alpha = \tau'\gamma$. Then $\tau'^2 = (\tau'\gamma)\gamma(\tau'\gamma)\gamma = (\alpha\gamma)(\alpha\gamma) = \mathrm{id}_R$.

(v)$\Rightarrow$(ii). Similar to the proof of (iv) $\Rightarrow$(ii). $\square$

We will introduce two examples of reversible automorphisms.

Example 2.1.3. Let $R = \mathbb{F}[t]$. Let α and γ be the $\mathbb{F}$-automorphisms such that $\alpha(t) = t + 1$ and $\gamma(t) = -t$. Then $\gamma^2 = \mathrm{id}_R$ and $\gamma\alpha(t) = 1 - t$. Thus $(\gamma\alpha)\gamma\alpha(t) = \gamma\alpha(1 - t) = \gamma(1 - (t - 1)) = t$, whence $(\gamma\alpha)^2 = \mathrm{id}_R$ and α is γ-reversible by Proposition 2.1.2.

Remark 2.1.4. In Example 2.1.3, the skew polynomial ring $R[x; \alpha]$ is generated by two elements x and t subject to the relation $xt - tx = x$. This is the enveloping algebra of the two-dimensional non-abelian solvable Lie algebra, and the skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ is its localization at the powers of the normal element x.

Example 2.1.5. Let $R = \mathbb{F}[t^{\pm 1}]$. Let α and γ be the $\mathbb{F}$-automorphisms such that $\alpha(t) = qt$, where $q \in \mathbb{F}\backslash\{0\}$, and $\gamma(t) = t^{-1}$. Then $\gamma^2 = \mathrm{id}_R$, $\gamma\alpha(t) = qt^{-1}$ and $(\gamma\alpha)\gamma\alpha(t) = \gamma\alpha(qt^{-1}) = q\gamma(q^{-1}t^{-1}) = t$. Thus $(\gamma\alpha)^2 = \mathrm{id}_R$ and α is γ-reversible by Proposition 2.1.2. Here the skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ is the quantum torus, see Definition 1.3.12.

Remark 2.1.6. If $\mathrm{char}\,\mathbb{F} = 0$ in Example 2.1.3 and Remark 2.1.4 then the skew Laurent polynomial ring $R[x^{\pm 1}; \alpha]$ is simple by Proposition 1.3.15. In Example 2.1.5, if q is not a root of unity in then $\mathcal{O}_q((\mathbb{F}^\times)^2)$ is a simple by Corollary 1.3.8.

Theorem 2.1.7. Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$. Let $S = R[x^{\pm 1}; \alpha]$. There exists $\theta \in \operatorname{Aut} S$ such that $\theta|_R = \gamma$ and $\theta(x) = x^{-1}$ if and only if α is γ-reversible.

Proof. Suppose that an automorphism θ with the stated properties exists. For each $r \in R$, $xr = \alpha(r)x$ and $x^{-1}r = \alpha^{-1}(r)x^{-1}$. Applying θ to the first of these, $x^{-1}\gamma(r) = \gamma\alpha(r)x^{-1}$. By the second, we see that $x^{-1}\gamma(r) = \alpha^{-1}\gamma(r)x^{-1}$, whence $\alpha^{-1}\gamma = \gamma\alpha$ and $\alpha^{-1} = \gamma\alpha\gamma$. Thus α is γ-reversible.

Conversely, suppose that α is γ-reversible and let $\phi = \iota\gamma : R \to S$, where ι is the embedding of R in S. The unit x^{-1} in the overring S of R is such that $x^{-1}\phi(r) = \alpha^{-1}\gamma(r)x^{-1} = \gamma\alpha(r)x^{-1} = \phi\alpha(r)x^{-1}$. By the universal mapping property for skew Laurent polynomial rings, as specified in Proposition 1.3.10, there is a (unique) ring endomorphism θ of S such that $\theta|_R = \gamma$ and $\theta(x) = x^{-1}$. As $\theta^2(r) = \gamma^2(r) = r$ for all $r \in R$ and $\theta^2(x) = x$, θ is its own inverse. Thus $\theta \in \operatorname{Aut} R$. $\square$

Here, let us make a definition.

Definition 2.1.8. A skew Laurent polynomial ring $S = R[x^{\pm 1}; \alpha]$ is **reversible** or $\gamma-$**reversible** if α is $\gamma-$reversible for some $\gamma \in \operatorname{Aut} R$ with $\gamma^2 = \mathrm{id}_R$. If S is reversible, then an automorphism θ of S such that $\theta(x) = x^{-1}$ and $\theta_R = \gamma$ will be called a **reversing** or $\gamma-$**reversing** automorphism of S.

Remark 2.1.9. In Example 2.1.3, we have shown that α is γ-reversible and, by Proposition 2.1.7, γ can be extended to a reversing automorphism θ of S such that $\theta(x) = x^{-1}$ and $\theta(t) = \gamma(t) = -t$. Observe that $tx^{-1} - x^{-1}t = \theta(xt - tx) = \theta(x) = x^{-1}$, and that, on applying θ, the relation $xt - tx = x$ becomes $tx^{-1} - x^{-1}t = x^{-1}$. In Example 2.1.5, by Proposition 2.1.7, the reversing automorphism θ is such that $\theta(t) = \gamma(t) = t^{-1}$ and $\theta(x) = x^{-1}$. It can be checked from $xt = qtx$ that $x^{-1}xtx^{-1} = qx^{-1}txx^{-1} \Rightarrow t^{-1}tx^{-1}t^{-1} = qt^{-1}x^{-1}tt^{-1} \Rightarrow x^{-1}t^{-1} = qt^{-1}x^{-1}$. Thus the relation $xt = qtx$ implies $x^{-1}t^{-1} = qt^{-1}x^{-1}$.

The next lemma identifies some elements of S^θ and some relations between them.

Notation 2.1.10. Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$ and α is γ-reversible. For $r \in R$ and $n \geq 0$, let $s_n(r) = rx^n + \gamma(r)x^{-n}$. In particular $s_0(r) = r + \gamma(r)$. Note that if R is an $\mathbb{F}-$algebra and $\alpha, \gamma \in \operatorname{Aut} R$ then the maps $s_n : R \to S^\theta$ are $\mathbb{F}$-linear, since for $r, r' \in R$,
$s_n(r + r') = (r + r')x^n + \gamma(r + r')x^{-n} = (rx^n + \gamma(r)x^{-n}) + (r'x^n + \gamma(r')x^{-n}) = s_n(r) + s_n(r')$. Also $s_n(\lambda r) = \lambda s_n(r)$ for all $r \in R$, $\lambda \in \mathbb{F}$.

Lemma 2.1.11. *Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$ and α is γ-reversible. Let $S = R[x^{\pm 1}; \alpha]$ and let $\theta \in \mathrm{Aut}(S)$ be such that $\theta(x) = x^{-1}$ and $\theta|_R = \gamma$. Then $s_n(r) \in S^\theta$. If $r, r' \in R$ are such that $rr' = r'r$ and $r' \in Z(R)$ then*

$$s_0(r)s_1(r') - s_1(r')s_0(\alpha^{-1}(r)) = s_1((\gamma(r) - \alpha^2\gamma(r))r'). \tag{2.1}$$

In particular, with $r' = 1$,

$$s_0(r)s_1(1) - s_1(1)s_0(\alpha^{-1}(r)) = s_1(\gamma(r) - \alpha^2\gamma(r)). \tag{2.2}$$

Proof. For $r \in R$ and $n \geq 0$, by Notation 2.1.10, $s_n(r) = rx^n + \gamma(r)x^{-n} \in S$. Observe that $\theta(s_n(r)) = \theta(rx^n + \gamma(r)x^{-n}) = \gamma(r)x^{-n} + \gamma^2(r)x^n = \gamma(r)x^{-n} + rx^n = s_n(r)$. Thus $s_n(r) \in S^\theta$.

Note that $\alpha\gamma\alpha^{-1} = \alpha\gamma(\gamma\alpha\gamma) = \alpha^2\gamma = \gamma\alpha^{-2}$. Also $\alpha^{-1}\gamma\alpha^{-1} = \gamma\alpha\alpha^{-1} = \gamma$. For $r,\ r' \in R$, we first show that

$$\begin{aligned}
& s_0(r)s_1(r') - s_1(r')s_0(\alpha^{-1}(r)) \\
= \;& (r + \gamma(r))(r'x + \gamma(r')x^{-1}) - (r'x + \gamma(r')x^{-1})(\alpha^{-1}(r) + \gamma\alpha^{-1}(r)) \\
= \;& (r + \gamma(r))r'x + (r + \gamma(r))\gamma(r')x^{-1} - r'x(\alpha^{-1}(r) + \gamma\alpha^{-1}(r)) \\
& -\gamma(r')x^{-1}(\alpha^{-1}(r) + \gamma\alpha^{-1}(r)) \\
= \;& rr'x + \gamma(r)r'x + r\gamma(r')x^{-1} + \gamma(rr')x^{-1} - r'rx - r'\alpha\gamma\alpha^{-1}(r)x \\
& -\gamma(r')\alpha^{-2}(r)x^{-1} - \gamma(r')\alpha^{-1}\gamma\alpha^{-1}(r)x^{-1} \\
= \;& \gamma(r)r'x + r\gamma(r')x^{-1} + \gamma(rr')x^{-1} - r'\alpha^2\gamma(r)x - \gamma(r')\gamma\alpha^2\gamma^{-1}(r)x^{-1} - \gamma(r'r)x^{-1} \\
= \;& s_1(\gamma(r)r' - r'\alpha^2\gamma(r)) = s_1((\gamma(r) - \alpha^2\gamma(r))r').
\end{aligned}$$

Now consider the case when $r' = 1$. By replacing $r' = 1$ in (2.1), we obtain $s_0(r)s_1(1) - s_1(1)s_0(\alpha^{-1}(r)) = s_1(\gamma(r) - \alpha^2\gamma(r))$. □

Proposition 2.1.12. *Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$ and α is γ-reversible. Let $S = R[x^{\pm 1}; \alpha]$ and let $\theta \in \mathrm{Aut}(S)$ be such that $\theta(x) = x^{-1}$ and $\theta|_R = \gamma$. Then the fixed ring S^θ is generated by R^γ and the set $\{s_1(r) : r \in R\}$.*

Proof. Let S_1 be the subring of S generated by R^γ and $\{s_1(r) : r \in R\}$. It is clear that $S_1 \subseteq S^\theta$. Let $s = \sum_{i=m}^n a_i x^i \in S^\theta$. Then $s = \theta(s) = \sum_{i=m}^n \gamma(a_i)x^{-i}$ from which it follows that $m = -n$, $a_0 = \gamma(a_0)$ and, for $1 \leq i \leq n$, $a_{-i} = \gamma(a_i)$. Thus $s = a_0 + \sum_{i=1}^n (a_i x^i + a_{-i}x^{-i}) = a_0 + \sum_{i=1}^n (a_i x^i + \gamma(a_i)x^{-i}) = a_0 + \sum_{i=1}^n s_i(a_i)$. Here we see that $a_0 \in R^\gamma \subseteq S_1$. It now suffices to show that for all $r \in R$ and all $i \geq 1$, $s_i(r) \in S_1$. We prove this result by induction on i. In the case in which $i = 1$ there is nothing to prove, since $s_1(r) \in S_1$. So we suppose, inductively, that $i \geq 1$ and that $s_j(r) \in S_1$ for all j, $1 \leq j \leq i$. Observe that

$$\begin{aligned} s_i(r)s_1(1) &= (rx^i + \gamma(r)x^{-i})(x + x^{-1}) \\ &= rx^{i+1} + \gamma(r)x^{-i+1} + rx^{i-1} + \gamma(r)x^{-(i+1)} \\ &= s_{i+1}(r) + s_{i-1}(r), \end{aligned}$$

whence $s_{i+1}(r) = s_i(r)s_1(1) - s_{i-1}(r) \in S_1$. This completes the proof. □

For the next result, we identify conditions under which S^θ is generated by R^γ and the single element $x + x^{-1}$.

Proposition 2.1.13. *Let R be a ring and let α, γ be automorphisms of R such that $\gamma^2 = \mathrm{id}_R$ and α is γ-reversible. Let $S = R[x^{\pm 1}; \alpha]$ and let $\theta \in \mathrm{Aut}(S)$ be such that $\theta(x) = x^{-1}$ and $\theta|_R = \gamma$. Let δ be the left α^2-derivation of R such that $\delta(r) = r - \alpha^2(r)$ for all $r \in R$. If $\delta(R) = R$ or if R is an $\mathbb{F}$-algebra, $\gamma \in \mathrm{Aut}_{\mathbb{F}} R$ and $\mathbb{F} + \delta(R) = R$ then S^θ is generated by R^γ and the element $s_1(1) = x + x^{-1}$.*

Proof. Let S_2 denote the subring of S^θ generated by R^γ and $x+x^{-1}$. By Proposition 2.1.12, it is enough to show that $s_1(r) \in S_2$ for all $r \in R$. Note that $s_0(r) = r + \gamma(r) \in R^\gamma$ for all $r \in R$.

Suppose that $\delta(R) = R$. As $\delta\gamma(R) = \delta(R) = R$, so $r = \delta\gamma(r_1)$ for some $r_1 \in R$ and, by (2.2),

$$s_1(r) = s_1(\gamma(r_1) - \alpha^2\gamma(r_1)) = s_0(r_1)s_1(1) - s_1(1)s_0(\alpha^{-1}(r_1)) \in S_2.$$

By Proposition 2.1.12, $S_2 = S^\theta$.

In the case where $\mathbb{F}+\delta(R) = R$, so $r = \lambda+\delta\gamma(r_2)$ for some $\lambda \in \mathbb{F}$ and $r_2 \in R$. It follows again from (2.2) that $s_1(\delta\gamma(r_2)) \in S_2$ and clearly $\lambda x+\gamma(\lambda)x^{-1} = \lambda s_1(1) \in S_2$ so

$$s_1(r) = s_1(\lambda + \delta\gamma(r_2)) = \lambda s_1(1) + s_1(\delta\gamma(r_2)) \in S_2.$$

Therefore $S_2 = S^\theta$. □

Corollary 2.1.14. *Let $S = \mathbb{F}[t][x^{\pm 1};\alpha]$ be as in Example 2.1.3 and $\theta \in \operatorname{Aut}_{\mathbb{F}} S$ be such that $\theta(t) = -t$ and $\theta(x) = x^{-1}$. Suppose that $\operatorname{char}\mathbb{F} = 0$. Then S^θ is generated by t^2 and $x+x^{-1}$.*

Proof. It is clear that $\mathbb{F}[t]^\gamma = \mathbb{F}[t^2]$ so, by Proposition 2.1.13, it suffices to show that $\delta(\mathbb{F}[t]) = \mathbb{F}[t]$. For $f \in \mathbb{F}[t]$, $\delta(f) = f - \alpha^2(f)$. For $n \geq 1$, $\delta(t^n) = t^n - (t+2)^n = g_n$ where $g_n \in \mathbb{F}[t]$ has degree $n-1$ and leading coefficient $-2n$. We will prove that $t^n \in \delta(\mathbb{F}[t])$ by induction on n. In the case in which $n = 0$, $\delta(t) = t - \alpha^2(t) = -2$ so $1 = \delta(-\frac{1}{2}t) \in \delta(\mathbb{F}[t])$. So we suppose, inductively, that $n \geq 0$ and that $t^j \in \delta(\mathbb{F}[t])$ for $0 \leq j \leq n$. Observe that

$$\delta(t^{n+1}) \quad = \quad t^{n+1} - (t+2)^{n+1} = -2(n+1)t^n + [\text{ lower terms}].$$

This implies that $2(n+1)t^n \in \delta(\mathbb{F}[t])$. Hence $\delta(\mathbb{F}[t]) = \mathbb{F}[t]$. □

Corollary 2.1.15. *Let $S = \mathbb{F}[t^{\pm 1}][x^{\pm 1};\alpha]$ be as in Example 2.1.5 and $\theta \in \operatorname{Aut} S$ be such that $\theta(t) = t^{-1}$ and $\theta(x) = x^{-1}$. Suppose that $\operatorname{char}\mathbb{F} \neq 2$ and that q is not a root of unity. Then S^θ is generated by $t+t^{-1}$ and $x+x^{-1}$.*

Proof. We can apply Proposition 2.1.12, with R replaced by $\mathbb{F}$ and α and γ replaced by $\mathrm{id}_{\mathbb{F}}$, so that θ is the $\mathbb{F}$-automorphism of $\mathbb{F}[x^{\pm 1}]$ such that $\theta(x) = x^{-1}$, to see that $\mathbb{F}[x^{\pm 1}]^{\theta} = \mathbb{F}[x + x^{-1}]$. Here the condition $R = \mathbb{F} + \delta(R)$ reduces to $\mathbb{F} = \mathbb{F}$. So, in the present notation, $\mathbb{F}[t^{\pm 1}]^{\gamma} = \mathbb{F}[t + t^{-1}]$. By Proposition 2.1.13, it suffices to show that $\mathbb{F} + \delta(\mathbb{F}[t^{\pm 1}]) = \mathbb{F}[t^{\pm 1}]$ where, for $f \in \mathbb{F}[t^{\pm 1}]$, $\delta(f) = f - \alpha^2(f)$. For $i \in \mathbb{Z}$, $\delta(t^i) = (1 - q^{2i})t^i$ where q is not a root of unity. If $i \neq 0$, then $t^i = \delta(\frac{1}{1-q^{2i}} t^i) \in \delta(\mathbb{F}[t^{\pm 1}])$. Hence $\mathbb{F} + \delta(\mathbb{F}[t^{\pm 1}]) = \mathbb{F}[t^{\pm 1}]$. $\square$

Remark 2.1.16. We have identified generators for S^{θ} for our two principal examples. In the next two sections, we shall identify a full set of defining relations, assuming that $\operatorname{char} \mathbb{F} = 0$ in Example 2.1.3 and that $\operatorname{char} \mathbb{F} \neq 2$ and q is not a root of unity in Example 2.1.5. For both examples, S^{θ} is simple by Theorem 1.2.3.

2.2 Invariants for the localized enveloping algebra

Throughout this section, $R = \mathbb{F}[t]$, $\alpha(t) = t + 1$, $\gamma(t) = -t$ and $\operatorname{char} \mathbb{F} = 0$. Thus $S = \mathbb{F}[t, x^{\pm 1} : xt - tx = x]$ and θ is the $\mathbb{F}$-automorphism of S such that $\theta(t) = -t$ and $\theta(x) = x^{-1}$. By Corollary 2.1.14 and Remark 2.1.16, S^{θ} is a simple $\mathbb{F}$-algebra and is generated by $u := t^2$ and $v := x + x^{-1}$. Our aim is to determine a set of defining relations for S^{θ}. We shall use a third "generator" $w := tx - tx^{-1}$. We shall identify S^{θ} as a homomorphic image T/pT where T is an iterated skew polynomial ring over $\mathbb{F}$ in three generators U, V and W and p is a central element of T of total degree 3. The extra generators w and W are used to get a PBW-basis for T.

We find the relations of the ring of invariants of S, S^{θ} as follows. Firstly,

$$\begin{aligned} vu &= (x + x^{-1})t^2 = (t+1)^2 x + (t-1)^2 x^{-1} \\ &= (t^2 + 2t + 1)x + (t^2 - 2t + 1)x^{-1} \\ &= t^2(x + x^{-1}) + 2(tx - tx^{-1}) + (x + x^{-1}) \\ &= uv + 2w + v. \end{aligned}$$

Hence

$$vu = uv + 2w + v. \tag{2.3}$$

The second relation can be checked as follows:

$$wv = (tx - tx^{-1})(x + x^{-1}) = tx^2 - tx^{-2}, \tag{2.4}$$

$$\begin{aligned} vw &= (x + x^{-1})(tx - tx^{-1}) = xtx - xtx^{-1} + x^{-1}tx - x^{-1}tx^{-1} \\ &= (t+1)x^2 - (t+1) + (t-1) - (t-1)x^{-2} \\ &= tx^2 - tx^{-2} + x^2 + x^{-2} - 2, \end{aligned} \tag{2.5}$$

and

$$v^2 = (x + x^{-1})^2 = x^2 + x^{-2} + 2. \tag{2.6}$$

Hence

$$wv = vw - v^2 + 4. \tag{2.7}$$

Also $uw = t^2(tx - tx^{-1})$ and

$$\begin{aligned} wu &= (tx - tx^{-1})t^2 = t(t+1)^2x - t(t-1)^2x^{-1} \\ &= t(t^2 + 2t + 1)x - t(t^2 - 2t + 1)x^{-1} \\ &= t^2(tx - tx^{-1}) + 2t^2(x + x^{-1}) + (tx - tx^{-1}) \\ &= uw + 2uv + w. \end{aligned} \tag{2.8}$$

Thus, by (2.3)

$$uw = (w - 2v)u + 3w + 2v. \tag{2.9}$$

We now use (2.3), (2.7) and (2.9) as a guide to constructing the iterated skew polynomial ring T in U, V and W mentioned above. Let $T_1 = \mathbb{F}[V]$ and let δ_1 be the $\mathbb{F}$-derivation $(4 - V^2)d/dV$ of T_1. Let $T_2 = T_1[W; \delta_1]$ so that, in accordance with (2.7), $WV = VW - V^2 + 4$. If F_2 is the free algebra $\mathbb{F}\langle \mathcal{V}, \mathcal{W} \rangle$ then $T_2 \simeq F_2/I$, where I is the ideal generated by $\mathcal{F} := \mathcal{W}\mathcal{V} - \mathcal{V}\mathcal{W} + \mathcal{V}^2 - 4$. There exists $\tau \in \operatorname{Aut}_{\mathbb{F}} F_2$ such that $\tau(\mathcal{V}) = \mathcal{V}$ and $\tau(\mathcal{W}) = \mathcal{W} - 2\mathcal{V}$, its inverse being such that $\tau^{-1}(\mathcal{V}) = \mathcal{V}$ and $\tau^{-1}(\mathcal{W}) = \mathcal{W} + 2\mathcal{V}$. It can be checked that

$$\begin{aligned} \tau(\mathcal{F}) &= \tau(\mathcal{W})\tau(\mathcal{V}) - \tau(\mathcal{V})\tau(\mathcal{W}) + \tau(\mathcal{V}^2) - 4 \\ &= (\mathcal{W} - 2\mathcal{V})\mathcal{V} - \mathcal{V}(\mathcal{W} - 2\mathcal{V}) + \mathcal{V}^2 - 4 \\ &= \mathcal{W}\mathcal{V} + \mathcal{V}^2 - \mathcal{V}\mathcal{W} - 4 = \mathcal{F}, \end{aligned}$$

whence $\tau(I) = I$ and there is an induced $\mathbb{F}$-automorphism σ of T_2 such that $\sigma(V) = V$ and $\sigma(W) = W - 2V$.

A left τ-derivation ∂ of F_2 is determined by specifying $\partial(\mathcal{V})$ and $\partial(\mathcal{W})$ and using the definition of a left τ-derivation in Definition 1.3.2 to extend to arbitrary elements of F_2. Here we set $\partial(\mathcal{V}) = -\mathcal{V} - 2\mathcal{W}$ and $\partial(\mathcal{W}) = 2\mathcal{V} + 3\mathcal{W}$. We then find that

$$\begin{aligned}
\partial(\mathcal{W}\mathcal{V}) &= \partial(\mathcal{W})\mathcal{V} + \tau(\mathcal{W})\partial\mathcal{V} \\
&= (2\mathcal{V} + 3\mathcal{W})\mathcal{V} + (\mathcal{W} - 2\mathcal{V})(-\mathcal{V} - 2\mathcal{W}) \\
&= 2\mathcal{V}^2 + 3\mathcal{W}\mathcal{V} - \mathcal{W}\mathcal{V} + 2\mathcal{V}^2 - 2\mathcal{W}^2 + 4\mathcal{V}\mathcal{W} \\
&= 2\mathcal{W}\mathcal{V} + 4\mathcal{V}^2 - 2\mathcal{W}^2 + 4\mathcal{V}\mathcal{W},
\end{aligned}$$

$$\begin{aligned}
\partial(\mathcal{V}\mathcal{W}) &= \partial(\mathcal{V})\mathcal{W} + \tau(\mathcal{V})\partial(\mathcal{W}) \\
&= (-\mathcal{V} - 2\mathcal{W})\mathcal{W} + \mathcal{V}(2\mathcal{V} + 3\mathcal{W}) \\
&= -\mathcal{V}\mathcal{W} - 2\mathcal{W}^2 + 2\mathcal{V}^2 + 3\mathcal{V}\mathcal{W} \\
&= 2\mathcal{V}\mathcal{W} + 2\mathcal{V}^2 - 2\mathcal{W}^2,
\end{aligned}$$

and

$$\begin{aligned}
\partial(\mathcal{V}^2) &= \partial(\mathcal{V})\mathcal{V} + \tau(\mathcal{V})\partial(\mathcal{V}) \\
&= (-\mathcal{V} - 2\mathcal{W})\mathcal{V} + \mathcal{V}(-\mathcal{V} - 2\mathcal{W}) \\
&= -\mathcal{V}^2 - 2\mathcal{W}\mathcal{V} - \mathcal{V}^2 - 2\mathcal{V}\mathcal{W} \\
&= -2\mathcal{W}\mathcal{V} - 2\mathcal{V}\mathcal{W} - 2\mathcal{V}^2
\end{aligned}$$

Thus $\partial(\mathcal{F}) = 0$. Thus $\partial(I) \subseteq I$ and there is an induced σ-derivation δ_2 of T_2 such that $\delta_2(V) = -V - 2W$ and $\delta_2(W) = 2V + 3W$. Let $T = T_2[U; \sigma, \delta_2]$. Then T is generated by U, V, W subject to the relations

$$\begin{aligned}
VU &= UV + 2W + V \\
WV &= VW - V^2 + 4 \\
UW &= (W - 2V)U + 3W + 2V.
\end{aligned}$$

By Proposition 1.3.11, therefore there is a surjective $\mathbb{F}$-homomorphism $\phi : T \to S^{\theta}$, such that $\phi(U) = u$, $\phi(V) = v$, $\phi(W) = w$ and $S^{\theta} \simeq T/\ker\phi$.

It remains to identify $\ker\phi$. Gelfand-Kirillov dimension is useful in doing so and is the key tool in the proof of the next result.

Proposition 2.2.1. *In the above situation,* $\ker\phi$ *is a prime ideal of* T *of height one.*

Proof. It is a consequence of Proposition 1.5.8 that $GK\dim(T_2) = 2$. Then apply Theorem 1.5.7 to $T = T_2[U;\sigma,\delta_2]$, $A = T_2, \alpha = \sigma, \delta = \delta_2$ and $B = \mathrm{Sp}(1, V, W)$. Since $\delta_2(V) = -V - 2W$, $\delta_2(W) = 2V + 3W$ it implies that $\delta_2(B) \subseteq B \subseteq B^2$. Since $\sigma(V) = V$ and $\sigma(W) = W - 2V$, so $\sigma(B) \subseteq B$. Hence $GK\dim(T) = GK\dim(T_2) + 1 = 3$. It follows from Corollary 1.5.6 that $GK\dim(S) = 2$. By Corollary 1.2.2, S is finitely generated as a right module over S^{θ}, so by Proposition 1.5.3, $GK\dim(S^{\theta}) = 2$. Hence $\ker\phi \neq 0$. Since $S^{\theta} \subseteq S$ is a domain, $\ker\phi$ is a prime ideal P, say, of T.

By Theorem 1.5.9, $3 = GK\dim(T) \geq GK\dim(T/P) + \mathrm{ht}(P) = GK\dim(S^{\theta}) + \mathrm{ht}(P) = 2 + \mathrm{ht}(P)$. Hence $\mathrm{ht}(P) \leq 1$. As T is a domain and $P = \ker\phi \neq 0$, $\mathrm{ht}(P) = 1$. $\square$

Proposition 2.2.2. *In the above ring* T*, let* $p = (4 - V^2)U + W^2 + 3VW + V^2 + 4$ *and let* $Z(T)$ *be the centre of* T*. Then* $p \in \ker\phi$*,* $p \in Z(T)$*,* pT *is a prime ideal of* T *and* $\ker\phi = pT$*.*

Proof. As T has been presented as an iterated skew polynomial ring $\mathbb{F}[V][W;\delta_1][U;\alpha,\delta_2]$, with coefficients on the left at each stage, we shall use the basis $\{V^iW^jU^k\}_{i,j,k\geq 0}$ for T. We calculate that

$$\begin{aligned} UV^2 &= (VU - V - 2W)V \\ &= VUV - V^2 - 2WV \\ &= V(VU - V - 2W) - V^2 - 2(VW - V^2 + 4) \\ &= V^2U - 4VW - 8. \end{aligned} \tag{2.10}$$

In S^θ,

$$\begin{aligned}
w^2 &= (tx - tx^{-1})(tx - tx^{-1}) \\
&= txtx - tx^{-1}tx - txtx^{-1} + tx^{-1}tx^{-1} \\
&= t(t+1)x^2 - t(t-1) - t(t+1) + t(t-1)x^{-2} \\
&= t^2(x^2 + x^{-2}) + (tx^2 - tx^{-2}) - 2t^2 \\
&= u(v^2 - 2) + wv - 2u \quad \text{by (2.4) and (2.6)} \\
&= v^2u - 4vw - 8 - 4u + wv \quad \text{by (2.10)} \\
&= v^2u - 3vw - 4u - v^2 - 4 \quad \text{by (2.7)}
\end{aligned}$$

so we have $w^2 - v^2u + 3vw + 4u + v^2 + 4 = 0$, whence $p \in \ker\phi$.

To check that $p \in Z(T)$, since $W = \frac{1}{2}(VU - UV - V)$ by (2.3), it is enough to show that $pU = Up$ and $pV = Vp$. We make the following calculations which use the U, V, W-analogues of (2.3), (2.7) and (2.9):

$$\begin{aligned}
UW^2 &= (W - 2V)UW + 3W^2 + 2VW \\
&= (W - 2V)(WU - 2VU + 3W + 2V) + 3W^2 + 2VW \\
&= W^2U - 2WVU + 3W^2 + 2WV - 2VWU + 4V^2U - 4VW - 4V^2 + 3W^2 \\
&= W^2U - 2(VW - V^2 + 4)U + 6W^2 + 2(VW - V^2 + 4) - 2VWU \\
&\quad +4V^2U - 4VW - 4V^2 \\
&= W^2U - 4VWU + 6V^2U - 8U + 6W^2 - 2VW - 6V^2 + 8, \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
W^2V &= W(VW - V^2 + 4) \\
&= (VW - V^2 + 4)W - (VW - V^2 + 4)V + 4W \\
&= VW^2 - V^2W + 8W - 4V + V^3 - V(VW - V^2 + 4) \\
&= VW^2 - 2V^2W + 2V^3 + 8W - 8V, \qquad (2.12)
\end{aligned}$$

and

$$\begin{aligned} UVW &= (VU - V - 2W)W \\ &= VUW - VW - 2W^2 \\ &= V(WU - 2VU + 3W + 2V) - VW - 2W^2 \\ &= VWU - 2V^2U + 2VW + 2V^2 - 2W^2. \end{aligned} \tag{2.13}$$

We first show that $Up = pU$. Using (2.11), (2.12) and (2.13),

$$\begin{aligned} Up &= U(W^2 - V^2U + 3VW + V^2 + 4U + 4) \\ &= UW^2 - UV^2U + 3UVW + UV^2 + 4U^2 + 4U \\ &= W^2U - 4VWU + 6V^2U - 8U + 6W^2 - 2VW - 6V^2 + 8 \\ &\quad -(V^2U^2 - 4VWU - 8U) + 3(VWU - 2V^2U + 2VW + 2V^2 - 2W^2) \\ &\quad +V^2U - 4VW - 8 + 4U^2 + 4U \\ &= W^2U - V^2U^2 + 3VWU + V^2U + 4U^2 + 4U \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} pU &= (W^2 - V^2U + 3VW + V^2 + 4U + 4)U \\ &= W^2U - V^2U^2 + 3VWU + V^2U + 4U^2 + 4U = Up. \end{aligned} \tag{2.15}$$

Finally, observe that

$$\begin{aligned} Vp &= V(W^2 - V^2U + 3VW + V^2 + 4U + 4) \\ &= VW^2 - V^3U + 3V^2W + V^3 + 4VU + 4V \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} pV &= (W^2 - V^2U + 3VW + V^2 + 4U + 4)V \\ &= W^2V - V^2UV + 3VWV + V^3 + 4UV + 4V \\ &= VW^2 - 2V^2W + 2V^3 + 8W - 8V - (V^3U - V^3 - 2V^2W) \\ &\quad +3V^2W - 3V^3 + 12V + 4VU - 4V - 8W + 4V + V^3 \\ &= VW^2 + 3V^2W - V^3U + 4VU + 4V + V^3 = Vp. \end{aligned} \tag{2.17}$$

Thus $pV = Vp$ and $p \in Z(T)$.

We next apply Proposition 1.4.5 to the central element p in the ring $T = T_1[U;\sigma,\delta_2]$. Writing $p = dU + e$ where $d = 4 - V^2$ and $e = W^2 + V^2 + 3VW + 4$, Proposition 1.4.5 says that d is a normal element and dT_1 is an ideal of T_1. To show that T/pT is a domain, we need to show that e is a non-zero divisor modulo dT_1. Note that $\overline{T_1} = T_1/dT_1 = \mathbb{F}[V,W]/(4 - V^2)$ is commutative, where $\mathbb{F}[V,W]$ is a commutative polynomial ring, and that every element of $\overline{T_1}$ has the unique form $f(W) + Vg(W)$ where $f(W)$, $g(W) \in \mathbb{F}[W]$ and $V^2 = 4$. Here we are abusing notation by writing V, W for their images in $\overline{T_1}$. We claim that $\overline{e} = e + dT_1 = 3VW + W^2 + 8$ is a non-zero divisor in $\overline{T_1}$. Assume that $\overline{e}t = 0$ for some $t = f(W)+Vg(W) \in \overline{T}$, that is, $(3VW+W^2+8)(f(W)+Vg(W)) = 0$. Then

$$3VWf(W) + (W^2 + 8)f(W) + 12Wg(W) + (W^2 + 8)Vg(W) = 0.$$

Here we have

$$(W^2 + 8)f(W) + 12Wg(W) = 0 \tag{2.18}$$

$$3Wf(W) + (W^2 + 8)g(W) = 0 \tag{2.19}$$

By multiplying (2.18) and (2.19) by W^2+8 and $12W$ respectively and subtracting, we obtain $(36W^2 - (W^2 + 8)^2)f(W) = 0$. Since $36W^2 - (W^2 + 8)^2 \neq 0$, it implies that $f(W) = 0$. Hence, by (2.18), $Wg(W) = 0$ so we get $g(W) = 0$. Hence $\overline{e}$ is not a zero-divisor in $\overline{T_1}$. By Proposition 1.4.5, T/pT is a domain and pT is a (completely) prime ideal of T.

By Proposition 2.2.1, $\ker\phi$ is a prime ideal of T of height one. As 0 and pT are prime and $0 \subset pT \subseteq \ker\phi$, it must be the case that $pT = \ker\phi$. □

The next result summarises the results of this section.

Theorem 2.2.3. *Let $S = \mathbb{F}[x^{\pm 1}, t : xt - tx = x]$ and θ be the $\mathbb{F}$-automorphism of S such that $\theta(t) = -t$ and $\theta(x) = x^{-1}$. The ring of invariants S^θ is isomorphic to T/pT, where T is the iterated skew polynomial ring $\mathbb{F}[V][W;\delta_1][U;\sigma,\delta_2]$ constructed from $\mathbb{F}$ using the derivation $\delta_1 = (4 - V^2)d/dV$ of $\mathbb{F}[V]$, the $\mathbb{F}$-automorphism σ*

of $\mathbb{F}[V][W;\delta_1]$ *such that* $\sigma(V) = V$ *and* $\sigma(W) = W - 2V$ *and the* σ*-derivation* δ_2 *of* $\mathbb{F}[V][W;\delta_1]$ *such that* $\delta_2(V) = -V - 2W$ *and* $\delta_2(W) = 2V + 3W$ *and* p *is the central element* $(4 - V^2)U + W^2 + 3VW + V^2 + 4$ *of* T*.*

Proof. This is immediate from the existence of a surjective homomorphism $\phi : T \to S^\theta$ such that $\ker\phi = pT$. $\square$

2.3 Invariants for the quantum torus

Throughout this section, $R = \mathbb{F}[t^{\pm 1}]$, $\alpha(t) = qt$, where $q \in \mathbb{F}\backslash\{0\}$ is not a root of unity, $\gamma(t) = t^{-1}$ and $\operatorname{char}\mathbb{F} \neq 2$. Thus $S = \mathbb{F}[t^{\pm 1}, x^{\pm 1} : xt = qtx]$ and θ is the $\mathbb{F}$-automorphism of S such that $\theta(t) = t^{-1}$ and $\theta(x) = x^{-1}$. By Corollary 2.1.15 and Remark 2.1.16, S^θ is a simple $\mathbb{F}$-algebra generated by $u := t + t^{-1}$ and $v := x + x^{-1}$. We shall again identify S^θ as a homomorphic image T_q/pT_q where T_q is an $\mathbb{F}$-algebra with $\mathrm{GKdim}(T_q) = 3$ and p is a central element. As in the previous section, we shall use a third "generator" $w := tx + t^{-1}x^{-1}$ to get a PBW-basis of T_q.

We begin by finding relations between the above generators u, v, w of the ring of invariants of S, S^θ, that is, $u = t + t^{-1}$, $v = x + x^{-1}$ and $w = tx + t^{-1}x^{-1}$. Firstly, observe that

$$\begin{aligned} uv &= (t + t^{-1})(x + x^{-1}) = tx + tx^{-1} + t^{-1}x + t^{-1}x^{-1} \\ &= (tx + t^{-1}x^{-1}) + (tx^{-1} + t^{-1}x) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} vu &= (x + x^{-1})(t + t^{-1}) \\ &= xt + xt^{-1} + x^{-1}t + x^{-1}t^{-1} \\ &= qtx + q^{-1}t^{-1}x + q^{-1}tx^{-1} + qt^{-1}x^{-1} \\ &= q(tx + t^{-1}x^{-1}) + q^{-1}(tx^{-1} + t^{-1}x). \end{aligned}$$

By (2.20), we obtain

$$uv - qvu = (1 - q^2)w. \tag{2.21}$$

Secondly, we find that

$$\begin{aligned} vw &= (x+x^{-1})(tx+t^{-1}x^{-1}) \\ &= xtx+xt^{-1}x^{-1}+x^{-1}tx+x^{-1}t^{-1}x^{-1} \\ &= qtx^2+q^{-1}t^{-1}+q^{-1}t+qt^{-1}x^{-2} \\ &= q^{-1}(t+t^{-1})+q(tx^2+t^{-1}x^{-2}) \\ &= q^{-1}u+q(tx^2+t^{-1}x^{-2}). \end{aligned}$$

Here

$$tx^2+t^{-1}x^{-2} = q^{-1}vw-q^{-2}u, \tag{2.22}$$

and

$$\begin{aligned} wv &= (tx+t^{-1}x^{-1})(x+x^{-1}) \\ &= tx^2+t+t^{-1}+t^{-1}x^{-2} \\ &= (tx^2+t^{-1}x^{-2})+(t+t^{-1}) \\ &= q^{-1}vw-q^{-2}u+u \quad \text{by (2.22)}, \end{aligned}$$

whence

$$vw-qwv = q^{-1}(1-q^2)u. \tag{2.23}$$

Finally, consider

$$\begin{aligned} uw &= (t+t^{-1})(tx+t^{-1}x^{-1}) \\ &= t^2x+x^{-1}+x+t^{-2}x^{-1} \\ &= (t^2x+t^{-2}x^{-1})+x^{-1}+x \\ &= (t^2x+t^{-2}x^{-1})+v \end{aligned} \tag{2.24}$$

and also

$$\begin{aligned} wu &= (tx+t^{-1}x^{-1})(t+t^{-1}) \\ &= txt+txt^{-1}+t^{-1}x^{-1}t+t^{-1}x^{-1}t^{-1} \\ &= qt^2x+q^{-1}x+q^{-1}x^{-1}+qt^{-2}x^{-1} \\ &= q(t^2x+t^{-2}x^{-1})+q^{-1}(x+x^{-1}). \end{aligned}$$

Thus by (2.24),

$$wu - quw = q^{-1}(1-q^2)v. \tag{2.25}$$

Let T_q denote the $\mathbb{F}$-algebra generated by $U,\ V,\ W$ subject to the relations

$$\begin{aligned} UV - qVU &= (1-q^2)W & (2.26)\\ VW - qWV &= q^{-1}(1-q^2)U & (2.27)\\ WU - qUW &= q^{-1}(1-q^2)V. & (2.28) \end{aligned}$$

Note that the relations (2.26), (2.27) and (2.28) in T_q have some symmetry which we increase by changing generators. Let

$$\mathcal{U} = \tfrac{q^{1/2}}{1-q^2}U, \quad \mathcal{V} = \tfrac{q^{1/2}}{1-q^2}V, \quad \mathcal{W} = \tfrac{q}{1-q^2}W, \tag{2.29}$$

where $q^{1/2}$ is one of the square roots of q in $\mathbb{F}$. Relations (2.26), (2.27) and (2.28) become

$$\begin{aligned} \mathcal{U}\mathcal{V} - q\mathcal{V}\mathcal{U} &= \mathcal{W}, & (2.30)\\ \mathcal{V}\mathcal{W} - q\mathcal{W}\mathcal{V} &= \mathcal{U}, & (2.31)\\ \mathcal{W}\mathcal{U} - q\mathcal{U}\mathcal{W} &= \mathcal{V}. & (2.32) \end{aligned}$$

For example, $\mathcal{U}\mathcal{V} - q\mathcal{V}\mathcal{U} = \frac{qUV - q^2VU}{(1-q^2)^2} = \frac{q}{1-q^2}W = \mathcal{W}$. As u, v and w generate S^θ, there is a surjective $\mathbb{F}$-homomorphism $\phi : T_q \to S^\theta$, such that $\phi(U) = u, \phi(V) = v$, $\phi(W) = w$ and $S^\theta \simeq T_q/\ker\phi$.

We are now going to identify $\ker\phi$ as in the previous section.

Proposition 2.3.1. *In the above situation,* $\ker\phi$ *is a prime ideal of* T_q *of height one.*

Proof. We have seen in Section 1.7, where the generator were written $\mathcal{U},\ \mathcal{V},\ \mathcal{W}$ that the irreducible monomials $\mathcal{W}^i\mathcal{V}^j\mathcal{U}^k$, $i, j, k \geq 0$ form a PBW-basis for T_q. We have also seen, in Example 1.7.5, that for the standard filtration of T_q for the degree function with $d(\mathcal{U}) = d(\mathcal{V}) = d(\mathcal{W}) = 1$, $\mathrm{gr}(T_q)$ is generated by X, Y, Z subject to the relations

$$XY = qYX, \quad YZ = qZY, \quad ZX = qXZ.$$

Where $X = \bar{\mathcal{U}}$, $Y = \bar{\mathcal{V}}$ and $Z = \bar{\mathcal{W}}$.

Here $\mathrm{gr}(T_q) = \mathbb{F}[Z][Y;\alpha_1][X;\alpha_2]$ is an iterated skew polynomial ring over $\mathbb{F}$, where α_1 and α_2 are the $\mathbb{F}$-automorphisms such that $\alpha_1(Z) = qZ$, $\alpha_2(Z) = q^{-1}Z$ and $\alpha_2(Y) = qY$. Therefore $\mathrm{gr}(T_q)$ is a domain so it follows, from Proposition 1.7.4, that T_q is a domain. It remains to show that $GK\dim(T_q) = 3$. Let B be the finite dimensional generating subspace $\mathrm{Sp}(1, \mathcal{U}, \mathcal{V}, \mathcal{W})$ and let B^n be the $\mathbb{F}-$subspace of T_q generated by the n-fold products of elements in B. Let V_n be the subspace spanned by the monomials $\{\mathcal{W}^i\mathcal{V}^j\mathcal{U}^k\}$ with $i,\ j,\ k \geq 0$ and $i + j + k \leq n$. Then $V_n \subseteq B^n$ and because no word appearing in (2.30) - (2.32) has degree > 2, $B^n \subseteq V_n$. The monomials $\{\mathcal{W}^i\mathcal{V}^j\mathcal{U}^k\}$ are linearly independent so T_q has the same $GK-$dimension as the commutative polynomial algebra $\mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{W}]$, that is, $GK\dim(T_q) = 3$.

For the same reasons as in the proof of Proposition 2.2.1, we know that $\mathrm{GKdim}(S^\theta) = 2$, $\ker\phi \neq 0$. Since S^θ is a domain and $S^\theta \simeq T_q/\ker\phi$, it follows that $\ker\phi$ is a (completely) prime ideal P, say. Thus by Theorem 1.5.9,

$$3 = GK\dim(T_q) \geq GK\dim(T_q/P) + \mathrm{ht}(P) = GK\dim(S^\theta) + \mathrm{ht}(P) = 2 + \mathrm{ht}(P).$$

Therefore $\ker\phi$ has height 1. $\square$

2.4 A central element of T_q

Proposition 2.4.1. *In the above ring T_q, let*

$$p_1 = WVU - qW^2 - q^{-2}V^2 - U^2 + 2(1 + q^{-2})$$

and let $Z(T_q)$ be the centre of T_q. Then $p_1 \in \ker\phi$ and $p_1 \in Z(T_q)$.

Proof. In S^θ, we have

$$u^2 = (t + t^{-1})^2 = t^2 + t^{-2} + 2 \tag{2.33}$$

$$v^2 = (x + x^{-1})^2 = x^2 + x^{-2} + 2 \tag{2.34}$$

and

$$\begin{aligned} w^2 &= (tx + t^{-1}x^{-1})^2 \\ &= txtx + txt^{-1}x^{-1} + t^{-1}x^{-1}tx + t^{-1}x^{-1}t^{-1}x^{-1} \\ &= qt^2x^2 + 2q^{-1} + qt^{-2}x^{-2} \\ &= q(t^2x^2 + t^{-2}x^{-2}) + 2q^{-1}. \qquad (2.35) \end{aligned}$$

Also

$$\begin{aligned} wvu &= (tx + t^{-1}x^{-1})(x + x^{-1})(t + t^{-1}) \\ &= (tx^2 + t + t^{-1} + t^{-1}x^{-2})(t + t^{-1}) \\ &= (tx^2 + t^{-1}x^{-2})(t + t^{-1}) + (t + t^{-1})^2 \\ &= tx^2t + tx^2t^{-1} + t^{-1}x^{-2}t + t^{-1}x^{-2}t^{-1} + u^2 \\ &= q^2(t^2x^2 + t^{-2}x^{-2}) + q^{-2}x^2 + q^{-2}x^{-2} + u^2 \\ &= q^2(q^{-1}w^2 - 2q^{-2}) + q^{-2}(v^2 - 2) + u^2 \quad \text{by (2.34) and (2.35)} \\ &= qw^2 + q^{-2}v^2 + u^2 - 2(1 + q^{-2}). \qquad (2.36) \end{aligned}$$

Hence $p_1 = WVU - qW^2 - q^{-2}V^2 - U^2 + 2(1 + q^{-2}) \in \ker \phi$.

It remains to show that $p_1 \in Z(T_q)$. By (2.28), $V = \frac{q}{1-q^2}(WU - qUW)$, so it is enough to show that $p_1W = Wp_1$ and $p_1U = Up_1$. Using (2.26), (2.27) and (2.28), we see that

$$\begin{aligned} UV^2 &= (qVU + (1 - q^2)W)V \\ &= qV(qVU + (1 - q^2)W) + (1 - q^2)WV \\ &= q^2V^2U + q(1 - q^2)VW + (1 - q^2)WV \\ &= q^2V^2U + q(1 - q^2)(qWV - (q - q^{-1})U) + (1 - q^2)WV \\ &= q^2V^2U + (1 - q^4)WV + (q^2 - 1)^2U, \qquad (2.37) \end{aligned}$$

$$\begin{aligned} UW^2 &= (q^{-1}WU + (1 - q^{-2})V)W \\ &= q^{-1}W(q^{-1}WU + (1 - q^{-2})V) + (1 - q^{-2})VW \\ &= q^{-2}W^2U + (q^{-1} - q^{-3})WV + (1 - q^{-2})(qWV - (q - q^{-1})U) \\ &= q^{-2}W^2U + (q - q^{-3})WV - q^{-1}(q - q^{-1})^2U, \qquad (2.38) \end{aligned}$$

$$\begin{aligned}
V^2W &= V(qWV-(q-q^{-1})U) \\
&= q(qWV-(q-q^{-1})U)V-(q-q^{-1})VU \\
&= q^2WV^2-(q^2-1)UV-(q-q^{-1})VU \\
&= q^2WV^2-(q^2-1)(qVU+(1-q^2)W)-(q-q^{-1})VU \\
&= q^2WV^2-(q^3-q^{-1})VU+(q^2-1)^2W, \qquad (2.39)
\end{aligned}$$

$$\begin{aligned}
U^2W &= U(q^{-1}WU+(1-q^{-2})V) \\
&= q^{-1}(q^{-1}WU+(1-q^{-2})V)U+(1-q^{-2})UV \\
&= q^{-2}WU^2+(q^{-1}-q^{-3})VU+(1-q^{-2})UV \\
&= q^{-2}WU^2+(q^{-1}-q^{-3})VU+(1-q^{-2})(qVU+(1-q^2)W) \\
&= q^{-2}WU^2+(q-q^{-3})VU-q^{-2}(1-q^2)^2W, \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
VUW &= V(q^{-1}WU+(1-q^{-2})V) \\
&= q^{-1}(qWV-(q-q^{-1})U)U+(1-q^{-2})V^2 \\
&= WVU-(1-q^{-2})U^2+(1-q^{-2})V^2, \qquad (2.41)
\end{aligned}$$

and

$$\begin{aligned}
UWV &= (q^{-1}WU+(1-q^{-2})V)V \\
&= q^{-1}WUV+(1-q^{-2})V^2 \\
&= q^{-1}W(qVU+(1-q^2)W)+(1-q^{-2})V^2 \\
&= WVU+(q^{-1}-q)W^2+(1-q^{-2})V^2. \qquad (2.42)
\end{aligned}$$

We first show that $Up_1 = p_1U$. Using (2.37), (2.38) and (2.42),

$$\begin{aligned}
Up_1 &= UWVU-qUW^2-q^{-2}UV^2-U^3+2(1+q^{-2})U \\
&= WVU^2+(q^{-1}-q)W^2U+(1-q^{-2})V^2U \\
&\quad -q(q^{-2}W^2U+(q-q^{-3})WV-q^{-1}(q-q^{-1})^2U) \\
&\quad -q^{-2}(q^2V^2U+(1-q^4)WV+(q^2-1)^2U)-U^3+2(1+q^{-2})U \\
&= WVU^2-qW^2U-q^{-2}V^2U-U^3+2(1+q^{-2})U = p_1U.
\end{aligned}$$

Next, let us observe that

$$Wp_1 = W^2VU - qW^3 - q^{-2}V^2W - U^2W + 2(1+q^{-2})W \tag{2.43}$$

and, using (2.39), (2.40) and (2.41), that

$$\begin{aligned} p_1W &= WVUW - qW^3 - q^{-2}V^2W - U^2W + 2(1+q^{-2})W \\ &= W(WVU + (q^{-1}-q)U^2 + (1-q^{-2})V^2) - q^{-2}(q^2WV^2 \\ &\quad -(q^3-q^{-1})VU + (q^2-1)^2W) - (q^{-2}WU^2 + (q-q^{-3})VU \\ &\quad -q^{-2}(1-q^2)^2W) - qW^3 + 2(1+q^{-2})W \\ &= W^2VU - WU^2 - q^{-2}WV^2 - qW^3 + 2(1+q^{-2})W = Wp_1. \end{aligned} \tag{2.44}$$

Thus $Wp_1 = p_1W$ by (2.43) and (2.44), whence $p_1 \in Z(T_q)$. □

Remark 2.4.2. With reference to Proposition 2.4.1 and ring T_q,

$$p_1 = WVU - qW^2 - q^{-2}V^2 - U^2 + 2(1+q^{-2})$$

is a central element of T_q. Using (2.29) with the generators $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, then

$$\begin{aligned} p_1 &= q^{-2}(1-q^2)^3\mathcal{W}\mathcal{V}\mathcal{U} - q^{-1}(1-q^2)^2\mathcal{W}^2 - q^{-3}(1-q^2)^2\mathcal{V}^2 \\ &\quad -q^{-1}(1-q^2)^2\mathcal{U}^2 + 2q^{-2}(1+q^2) \\ &= q^{-2}(1-q^2)^2\left((1-q^2)\mathcal{W}\mathcal{V}\mathcal{U} - q\mathcal{W}^2 - q^{-1}\mathcal{V}^2 - q\mathcal{U}^2 + 2(1+q^2)/(1-q^2)^2\right). \end{aligned}$$

Setting $p := (1-q^2)\mathcal{W}\mathcal{V}\mathcal{U} - q\mathcal{W}^2 - q^{-1}\mathcal{V}^2 - q\mathcal{U}^2 + 2(1+q^2)/(1-q^2)^2$. It follows from Proposition 2.4.1 that $p \in \ker\phi$ and $p \in Z(T_q)$.

Remark 2.4.3. We shall be interested in a homomorphic image of T_q, namely the $\mathbb{F}$-algebra B generated by X, Y and Z subject to the relations

$$\begin{aligned} XY - qYX &= Z \\ YZ - qZY &= X \\ ZX - qXZ &= Y \\ (1-q^2)ZYX &= qZ^2 + qX^2 + q^{-1}Y^2 + \kappa \end{aligned}$$

for some $\kappa \in \mathbb{F}$. If we form the associated graded ring for the standard filtration with $d(X) = d(Y) = d(Z) = 1$, the fourth defining equation leads to zero-divisors. It turns out to be more appropriate to use the degree function with $d(Y) = d(Z) = 1$ but $d(X) = 0$. The order of the generators for the lexicographic ordering is $X > Y > Z$.

Our next goal is to show T_q/pT_q is a domain.

2.5 Another associated graded ring $\mathrm{gr}(T_q)$

Remark 2.5.1. To show that T_q/pT_q is a domain, we shall use associated graded rings. If we form $\mathrm{gr}(T_q/pT_q)$ using the standard filtration for the same degree function as for T_q in Example 1.6.1, Example 1.7.5 and Proposition 2.3.1, we do not get a domain because $\overline{\mathcal{W}}\ \overline{\mathcal{V}}\ \overline{\mathcal{U}} = 0$. Therefore we consider the standard filtration of T_q for degree function with $d(\mathcal{V}) = d(\mathcal{W}) = 1$ but $d(\mathcal{U}) = 0$. The reduction system S for T_q in Example 1.6.1 is compatible with the $d-$length lexicographic ordering $\mathcal{U} > \mathcal{V} > \mathcal{W}$ which has DCC by Lemma 1.7.6. Therefore $\mathrm{gr}(T_q)$ has a PBW-basis obtained using the vector space isomorphism $\psi : T \to \mathrm{gr}(T_q)$ in Section 1.7. In $\mathrm{gr}(T_q)$,

$$\overline{\mathcal{U}}\ \overline{\mathcal{V}} = q\overline{\mathcal{V}}\ \overline{\mathcal{U}} + \overline{\mathcal{W}}, \quad \overline{\mathcal{V}}\ \overline{\mathcal{W}} = q\overline{\mathcal{W}}\ \overline{\mathcal{V}}, \quad \overline{\mathcal{W}}\ \overline{\mathcal{U}} - q\overline{\mathcal{U}}\ \overline{\mathcal{W}} = \overline{\mathcal{V}}.$$

To investigate $\mathrm{gr}(T_q)$, let C_q be the $\mathbb{F}$-algebra generated by $a,\ b,\ c$ subject to the relations

$$ab - qba = c, \quad bc = qcb, \quad ca - qac = b. \tag{2.45}$$

There is a surjective $\mathbb{F}-$homomorphism $\phi : C_q \to \mathrm{gr}(T_q)$ with $\phi(a) = \overline{\mathcal{U}},\ \phi(b) = \overline{\mathcal{V}}$ and $\phi(c) = \overline{\mathcal{W}}$. We use the same method as in Section 2.2, to show that C_q is an iterated skew polynomial ring over $\mathbb{F}$. To do this, let $D_1 = \mathbb{F}[c]$ and let α_1 be the $\mathbb{F}$-automorphism of D_1 such that $\alpha_1(c) = qc$. Let $D_2 = D_1[b; \alpha_1]$, the coordinate ring of the quantum plane with the relation $bc = qcb$. If F_2 is the free algebra $\mathbb{F}\langle b', c'\rangle$ then $D_2 \simeq F_2/I$, where I is the ideal generated by $J := b'c' - qc'b'$. There exists $\tau \in \mathrm{Aut}_{\mathbb{F}} F_2$ such that $\tau(b') = qb'$ and $\tau(c') = q^{-1}c'$ and this induces an $\mathbb{F}$-automorphism α_2 of D_2 such that $\alpha_2(b) = qb$ and $\alpha_2(b) = q^{-1}b$.

There is a left τ-derivation ∂ of F_2 such that $\partial(b') = c'$ and $\partial(c') = -q^{-1}b'$. By Definition 1.3.2, we find that $\partial(b'c') = \partial(b')c' + \tau(b')\partial(c') = c'^2 - b'^2$ and $\partial(qc'b') = q(\partial(c')b' + \tau(c')\partial(b')) = q(q^{-1}c'^2 - q^{-1}b'^2) = c'^2 - b'^2$, whence $\partial(J) = 0$. Therefore $\partial(I) \subseteq I$ and there is an induced α_2-derivation δ_2 of D_2 such that $\delta_2(b) = c$ and $\delta_2(c) = -q^{-1}c$. Let $D_3 = D_2[a; \alpha_2, \delta_2]$. By Proposition 1.3.11, D_3 is generated by $a,\ b,\ c$ subject to the relations (2.45) so $C_q \simeq D_3$ and hence C_q has a PBW-basis. It follows that ϕ is an isomorphism and $gr(T_q) \simeq C_q \simeq D_3$. Thus $\mathrm{gr}(T_q)$ is an iterated skew polynomial ring.

Proposition 2.5.2. *The algebra T_q/pT_q is a domain.*

Proof. To prove the Proposition, let us consider the standard filtration of T_q/pT_q for the degree function with $d(Y) = d(Z) = 1$ but $d(X) = 0$ where $X = \mathcal{U} + pT_q$, $Y = \mathcal{V} + pT_q$ and $Z = \mathcal{W} + pT_q$. Then $X,\ Y,\ Z$ satisfy the relations

$$\begin{aligned} XY &= qYX + Z, && (2.46)\\ YZ &= qZY + X, && (2.47)\\ XZ &= q^{-1}ZX - q^{-1}Y \quad \text{and} && (2.48)\\ Y^2 &= q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa, && (2.49)\end{aligned}$$

where $\kappa = 2(1+q^2)/(1-q^2)^2$. These have been written to conform to a reduction system S of T_q/pT_q that is compatible with the $d-$length lexicographic ordering $\leq$ and with the degree function d. By Lemma 1.7.6, $\leq$ has DCC so we can apply the Diamond Lemma 1.6.2. There are no inclusion ambiquities and the only two extra overlap ambiguities $(XY)Y = X(Y^2)$ and $(Y^2)Z = Y(YZ)$, in addition to $(XY)Z = X(YZ)$ which has already been resolved in Example 1.6.1. Before resolving these, we need some identities. Using (2.46), (2.47), (2.48), we see that

$$\begin{aligned} XZY &= (q^{-1}ZX - q^{-1}Y)Y\\ &= q^{-1}Z(qYX + Z) - q^{-1}Y^2\\ &= ZXY + q^{-1}Z^2 - q^{-1}(q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa)\\ &= q^2ZYX + (q^{-1}+q)Z^2 + qX^2 - \kappa, && (2.50)\end{aligned}$$

$$\begin{aligned} XZ^2 &= (q^{-1}ZX - q^{-1}Y)Z \\ &= q^{-1}Z(q^{-1}ZX - q^{-1}Y) - q^{-1}(qZY + X) \\ &= q^{-2}Z^2X - (1+q^{-2})ZY - q^{-1}X \end{aligned} \tag{2.51}$$

$$\begin{aligned} YXZ &= Y(q^{-1}ZX - q^{-1}Y) = q^{-1}(qZY + X)X - q^{-1}Y^2 \\ &= ZYX + q^{-1}X^2 - q^{-1}(q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa) \\ &= q^2ZYX + (q^{-1}+q)X^2 + qZ^2 - \kappa \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} X^2Z &= X(q^{-1}ZX - q^{-1}Y) \\ &= q^{-1}(q^{-1}ZX - q^{-1}Y)X - q^{-1}(qYX + Z) \\ &= q^{-2}ZX^2 - (1+q^{-2})YX - q^{-1}Z. \end{aligned} \tag{2.53}$$

To resolve the ambiguity $(XY)Y = X(Y^2)$, we note that

$$\begin{aligned} (XY)Y &= (qYX + Z)Y = qY(qYX + Z) + ZY \\ &= q^2(q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa)X + q(qZY + X) + ZY \\ &= q^3(1-q^2)ZYX^2 - q^4Z^2X - q^4X^3 + (1+q^2)ZY + (q+q^3\kappa)X \end{aligned}$$

and

$$\begin{aligned} XY^2 &= X(q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa) \\ &= q(1-q^2)XZYX - q^2XZ^2 - q^2X^3 + q\kappa X \\ &= q(1-q^2)(q^2ZYX + q^{-1}(1+q^2)Z^2 + qX^2 - \kappa)X - q^2X^3 + q\kappa X \\ &\quad -q^2(q^{-2}Z^2X - (1-q^{-2})ZY - q^{-1}X) \quad \text{by (2.50) and (2.51)} \\ &= q^3(1-q^2)ZYX^2 + (1-q^4)Z^2X - q^2(1-q^2)X^3 \\ &\quad -q(1-q^2)\kappa X - Z^2X + (1+q^2)ZY + qX - q^2X + q\kappa X \\ &= q^3(1-q^2)ZYX^2 - q^4Z^2X - q^4X^3 + (1+q^2)ZY + (q+q^3\kappa)X \\ &= (XY)Y. \end{aligned}$$

To resolve the ambiguity $Y(YZ) = (Y^2)Z$, we note that

$$\begin{aligned}
Y(YZ) &= Y(qZY + X) = q(qZY + X)Y + YX \\
&= q^2Z(q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa) + q(qYX + Z) + YX \\
&= q^3(1-q^2)Z^2YX - q^4Z^3 - q^4ZX^2 + q^3\kappa Z + (1+q^2)YX + qZ \\
&= q^3(1-q^2)Z^2YX - q^4Z^3 - q^4ZX^2 + (1+q^2)YX + (q+q^3\kappa)Z
\end{aligned}$$

and

$$\begin{aligned}
Y^2Z &= (q(1-q^2)ZYX - q^2Z^2 - q^2X^2 + q\kappa)Z \\
&= q(1-q^2)ZYXZ - q^2Z^3 - q^2X^2Z + q\kappa Z \\
&= q(1-q^2)Z(q^2ZYX + (q^{-1}+q)X^2 + qZ^2 - \kappa) - q^2Z^3 \\
&\quad -q^2(q^{-2}ZX^2 - (1+q^{-2})YX - q^{-1}Z) + q\kappa Z \quad \text{by (2.52) and (2.53)} \\
&= q^3(1-q^2)Z^2YX + (1-q^4)ZX^2 + q^2(1-q^2)Z^3 - (q-q^3)\kappa Z \\
&\quad -q^2Z^3 - ZX^2 + (1+q^2)YX + qZ + q\kappa Z \\
&= q^3(1-q^2)Z^2YX - q^4Z^3 - q^4ZX^2 + (1+q^2)YX + (q+q^3\kappa)Z \\
&= Y(YZ).
\end{aligned}$$

Therefore the Diamond Lemma 1.6.2 implies that the set

$$\{Z^iY^jX^k : i,j,k \geq 0, j < 2\}$$

is a basis for T_q/pT_q. Using the vector space isomorphism $\psi : T_q \to \mathrm{gr}(T_q)$ from Section 1.7, we see that the set $\{c^ib^ja^k : i,j,k \geq 0, j < 2\}$ form a basis for $\mathrm{gr}(T_q/pT_q)$.

Here $\mathrm{gr}(T_q/pT_q)$ is generated by $a = \overline{X}$, $b = \overline{Y}$ and $c = \overline{Z}$ and these satisfy the relations in (2.45) and also

$$(1-q^2)cba = qc^2 + q^{-1}b^2.$$

We claim that $\mathrm{gr}(T_q/pT_q) \simeq \mathrm{gr}(T_q)/\overline{p}\,\mathrm{gr}(T_q)$. It follows from Remark 2.5.1 that $\mathrm{gr}(T_q) \simeq D_3$ and also $\mathrm{gr}(T_q)/\overline{p}\,\mathrm{gr}(T_q) \simeq D_3/hD_3$, where h is the central element $(1-q^2)cba - qc^2 - q^{-1}b^2$ of D_3. Using the same semigroup ordering and degree function as for T_q/pT_q, but with a, b, c replacing X, Y, Z, the Diamond Lemma shows that $\{c^i b^j a^k : i, j, k \geq 0, j < 2\}$ is a basis for D_3/hD_3. The details of the calculations are the same except that some terms of low degree are deleted. It follows that $\mathrm{gr}(T_q/pT_q) \simeq D_3/hD_3$.

To show T_q/pT_q is a domain, we shall show that D_3/hD_3 is a domain and refer to Proposition 1.7.4. We apply Proposition 1.4.5 to the element h of $D_3 = D_2[a; \alpha_2, \delta_2]$. In the notation of Proposition 1.4.5, $R = D_3$, $A = D_2$, $c = h$, $d = cb$ and $e = q^2c^2 + b^2$. We want to show that e is a non-zero-divisor modulo cbD_2. Every element $t \in D_2/cbD_2$ can be written uniquely in the form $t = k + c\sigma(c) + b\beta(b) \in D_2/cbD_2$ where $k \in \mathbb{F}$, $\sigma(c) \in \mathbb{F}[c]$ and $\beta(b) \in \mathbb{F}[b]$. Here we are abusing notation and using b and c for their images in D_2/cbD_2. Suppose that $et = 0$, that is, $(b^2 + q^2c^2)(k + c\sigma(c) + b\beta(b)) = 0$. Then

$$\begin{aligned} k(b^2 + q^2c^2) + (b^2 + q^2c^2)c\sigma(c) + (b^2 + q^2c^2)b\beta(b) &= 0 \quad \text{and} \\ k(b^2 + q^2c^2) + q^2c^3\sigma(c) + b^3\beta(b) &= 0. \end{aligned}$$

It follows that $k = 0$ and $\sigma(c) = 0 = \beta(b)$. Similarly, $te = 0$ implies $t = 0$. Hence e is a non-zero regular modulo cbD_2, whence, by Proposition 1.4.5, D_3/hD_3 is a domain and so is $\mathrm{gr}(T_q/pT_q)$. By Proposition 1.7.4, T_q/pT_q is a domain. □

Theorem 2.5.3. *Let $S = \mathbb{F}[x^{\pm 1}, t^{\pm 1} : xt = qtx]$ and θ be the $\mathbb{F}$-automorphism of S such that $\theta(t) = t^{-1}$ and $\theta(x) = x^{-1}$. If q is not a root of unity then the ring of invariants S^θ is isomorphic to T_q/pT_q, where p is the central element*

$$(1 - q^2)\mathcal{W}\mathcal{V}\mathcal{U} - q\mathcal{W}^2 - q^{-1}\mathcal{V}^2 - q\mathcal{U}^2 + 2(1 + q^2)/(1 - q^2)^2$$

of T_q.

Proof. This is now immediate from Propositions 2.3.1, Proposition 2.4.1 and 2.5.2.

□

Chapter 3

Finite-dimensional simple T_q-modules

In this chapter we shall classify finite-dimensional simple modules over the $\mathbb{F}-$algebra T_q in Section 2.3 where q is not a root of unity in $\mathbb{F}$. For $n \geq 1$, each n-dimensional simple left T_q-modules gives rise to a Leonard triple as introduced by Terwilliger [22].

Throughout this chapter, $q \in \mathbb{F}\backslash\{0\}$ and is not a root of unity. We denote by T_q the $\mathbb{F}-$algebra generated by $x,\ y,\ z$ subject to the relations

$$xy - qyx = z, \tag{3.1}$$

$$yz - qzy = x, \tag{3.2}$$

$$zx - qxz = y. \tag{3.3}$$

3.1 Automorphisms

We shall give some automorphisms of T_q. Let σ be the $\mathbb{F}$-automorphism of T_q, with order 3, such that

$$\sigma(x) = y, \quad \sigma(y) = z, \quad \sigma(z) = x.$$

Then σ is cyclically permuting the generators of T_q. For each generator $v = x,\ y$ or z there is an $\mathbb{F}$-automorphism ϕ_v of T_q, with order 2, fixing v and multiplying the other two generators by -1. For example:

$$\phi_y(x) = -x, \quad \phi_y(y) = y, \quad \phi_y(z) = -z.$$

We shall classify finite-dimensional simple T_q-modules and start with the one-dimensional.

3.2 The one-dimensional simple modules

In this section, we shall classify one-dimensional simple modules by considering the relations (3.1)-(3.3). It is easy to check that $x = y = z = 0$ satisfy the relations (3.1)-(3.3). Thus there is a one-dimensional T_q-module $M = T_q/(T_q x + T_q y + T_q z)$.

We next find more one-dimensional left modules, where x, y, z do not act as zero, by changing the generators:

$$x' = (1-q)x, \quad y' = (1-q)y, \quad z' = (1-q)z. \tag{3.4}$$

Note that the relations (3.1), (3.2) and (3.3) become

$$x'y' - qy'x' = (1-q)z', \tag{3.5}$$

$$y'z' - qz'y' = (1-q)x', \tag{3.6}$$

$$z'x' - qx'z' = (1-q)y'. \tag{3.7}$$

For example, $x'y' - qy'x' = (1-q)^2 xy - q(1-q)^2 yx = (1-q)^2 z = (1-q)z'$.

Let V be Klein's four group: $V = \{1, x', y', z'\}$ with the relations $x'^2 = y'^2 = z'^2 = 1$ and $x'y' = z' = y'x'$. Let $\mathbb{F}V$ be the group algebra: $\mathbb{F}V = \mathbb{F} \oplus \mathbb{F}x' \oplus \mathbb{F}y' \oplus \mathbb{F}z'$. It is clear that x', y' and z' satisfy the relations (3.5)-(3.7). As x, y, z generate T_q, there exists a surjective $\mathbb{F}$-homomorphism $\phi : T_q \to \mathbb{F}V$ such that $\phi(x) = x'$, $\phi(y) = y'$ and $\phi(z) = z'$. Therefore each $\mathbb{F}V-$module M gives rise to a T_q-module by the rule $t \cdot m = \phi(t)m$, $t \in T_q$, $m \in M$. In particular the four simple $\mathbb{F}V-$modules, $\mathbb{F}V/(\mathbb{F}V(x' - \lambda_1) + \mathbb{F}V(y' - \lambda_2) + \mathbb{F}V(z' - \lambda_3))$, where $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$, and $\lambda_3 = \lambda_1 \lambda_2$, each give rise to a one-dimensional simple T_q-module.

Any one-dimensional left T_q-module has the form

$$M = T_q/(T_q(x' - \lambda_1) + T_q(y' - \lambda_2) + T_q(z' - \lambda_3))$$

where (3.5)-(3.7) are satisfied when $\lambda_1,\ \lambda_2,\ \lambda_3$ are substituted for $x',\ y'$ and z' respectively. The substitution gives

$$\lambda_1\lambda_2 = \lambda_3, \quad \lambda_2\lambda_3 = \lambda_1, \quad \lambda_1\lambda_3 = \lambda_2.$$

One solution is $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and if any $\lambda_i = 0$ then so are the other two so, in any other solution, $\lambda_i \neq 0$ for all i. It follows easily that $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = 1$. As $\lambda_3 = \lambda_1\lambda_2$, there are four non-zero solutions:

(i) $\lambda_1 = \lambda_2 = \lambda_3 = 1$
$M = T_q/(T_q(x' - 1) + T_q(y' - 1) + T_q(z' - 1))$
$= T_q/(T_q(x - \frac{1}{1-q}) + T_q(y - \frac{1}{1-q}) + T_q(z - \frac{1}{1-q}))$;

(ii) $\lambda_1 = \lambda_3 = 1,\ \lambda_2 = -1$
$M = T_q/(T_q(x' - 1) + T_q(y' + 1) + T_q(z' - 1))$
$= T_q/(T_q(x - \frac{1}{1-q}) + T_q(y + \frac{1}{1-q}) + T_q(z - \frac{1}{1-q}))$;

(iii) $\lambda_2 = \lambda_3 = 1,\ \lambda_1 = -1$
$M = T_q/(T_q(x' + 1) + T_q(y' - 1) + T_q(z' - 1))$
$= T_q/(T_q(x + \frac{1}{1-q}) + T_q(y - \frac{1}{1-q}) + T_q(z - \frac{1}{1-q}))$;

(iv) $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = -1$
$M = T_q/(T_q(x' - 1) + T_q(y' - 1) + T_q(z' + 1))$
$= T_q/(T_q(x - \frac{1}{1-q}) + T_q(y - \frac{1}{1-q}) + T_q(z + \frac{1}{1-q}))$.

Thus there are five one-dimensional simple left T_q-modules.

Notation 3.2.1. [16, p. 121] We introduce some notation. Let $q \in \mathbb{F}\backslash\{0\}$ and $q \neq \pm 1$. For any integer n, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q^{1-n}(q^{2n} - 1)}{q^2 - 1}. \tag{3.8}$$

Note that $[-n] = -[n]$ and $[m + n] = q^n[m] + q^{-m}[n]$. Observe that, if q is not a root of unity, then $[n] \neq 0$ for any non-zero integer.

3.3 Sequence of eigenvalues λ_i

Sequences of eigenvalues for the action of x are a key tool to identify finite-dimensional left T_q-modules. In this section we study the sequences that arise. Throughout $q \in \mathbb{F}\backslash\{0\}$ and is not a root of unity.

Lemma 3.3.1. *Let $\lambda_1, \lambda_2 \in \mathbb{F}$, not both 0, and let $\{\lambda_n\}_{n\geq 1}$ be the sequence defined by the recurrence relation*

$$\lambda_{n+2} = (q+q^{-1})\lambda_{n+1} - \lambda_n, \quad n \geq 1.$$

Then

$$\lambda_n = [n-1]\lambda_2 - [n-2]\lambda_1$$

for $n \geq 1$.

Proof. This uses the standard theory of linear recurrence relations. The defining equation for the relation can be expressed as

$$\begin{pmatrix} \lambda_{n+2} \\ \lambda_{n+1} \end{pmatrix} = A \begin{pmatrix} \lambda_{n+1} \\ \lambda_n \end{pmatrix} \text{ where } A = \begin{pmatrix} q+q^{-1} & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix A has distinct eigenvalues q and q^{-1} and $A = BDB^{-1}$ where

$$B = \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}, \text{ so that, } B^{-1} = (1-q^2)^{-1}\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}, \text{ and } D = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

Since $D^n = \begin{pmatrix} q^{-n} & 0 \\ 0 & q^n \end{pmatrix}$, we have $A^n = BD^nB^{-1}$, that is,

$$\begin{aligned} A^n &= (1-q^2)^{-1}\begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}\begin{pmatrix} q^{-n} & 0 \\ 0 & q^n \end{pmatrix}\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix} \\ &= (1-q^2)^{-1}\begin{pmatrix} q^{-n} & q^{n+1} \\ q^{1-n} & q^n \end{pmatrix}\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix} \\ &= (1-q^2)^{-1}\begin{pmatrix} q^{-n}-q^{n+2} & q^{n+1}-q^{1-n} \\ q^{1-n}-q^{1+n} & q^n-q^{2-n} \end{pmatrix}. \end{aligned}$$

Also,

$$\begin{pmatrix} \lambda_{n+2} \\ \lambda_{n+1} \end{pmatrix} = A^n \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} = \frac{1}{1-q^2} \begin{pmatrix} q^{-n} - q^{n+2} & q^{n+1} - q^{1-n} \\ q^{1-n} - q^{1+n} & q^n - q^{2-n} \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}.$$

Here

$$\begin{aligned} \begin{pmatrix} \lambda_{n+2} \\ \lambda_{n+1} \end{pmatrix} &= \frac{1}{1-q^2} \begin{pmatrix} (q^{-n} - q^{n+2})\lambda_2 + (q^{n+1} - q^{1-n})\lambda_1 \\ (q^{1-n} - q^{1+n})\lambda_2 + (q^n - q^{2-n})\lambda_1 \end{pmatrix} \\ &= \frac{1}{1-q^2} \begin{pmatrix} q^{-n}(1 - q^{2n+2})\lambda_2 - q^{n-1}(1 - q^{2n})\lambda_1 \\ q^{1-n}(1 - q^{2n})\lambda_2 - q^{2-n}(1 - q^{2n-2})\lambda_1 \end{pmatrix}. \end{aligned}$$

Thus

$$\lambda_{n+1} = [n]\lambda_2 - [n-1]\lambda_1$$

and also $\lambda_n = [n-1]\lambda_2 - [n-2]\lambda_1$ for $n \geq 1$. □

Note that we can extend the definition of λ_n, $n \geq 1$, to define λ_n for all $n \in \mathbb{Z}$, using the formula $\lambda_{n-2} = (q+q^{-1})\lambda_{n-1} - \lambda_n$. The recurrence formula in Lemma 3.3.1 is then true for all $n \in \mathbb{Z}$.

Remark 3.3.2. There are two alternative approaches to determining the sequences in Lemma 3.3.1. For $n \geq 1$, let

$$\delta_n = \lambda_{n+1} - q^{-1}\lambda_n, \quad \delta'_n = \lambda_{n+1} - q\lambda_n.$$

Then, for $n \geq 1$,

$$\delta_{n+1} = \lambda_{n+2} - q^{-1}\lambda_{n+1} = (q + q^{-1})\lambda_{n+1} - \lambda_n - q^{-1}\lambda_{n+1} = q\lambda_{n+1} - \lambda_n = q\delta_n$$

and

$$\delta'_{n+1} = \lambda_{n+2} - q\lambda_{n+1} = (q + q^{-1})\lambda_{n+1} - \lambda_n - q\lambda_{n+1} = q^{-1}\lambda_{n+1} - \lambda_n = q^{-1}\delta'_n,$$

whence $\delta_n = q^{n-1}\delta_1$ and $\delta'_n = q^{1-n}\delta'_1$. Thus

$$\lambda_{n+1} = q^{-1}\lambda_n + q^{n-1}\delta_1 = q\lambda_n + q^{1-n}\delta'_1. \tag{3.9}$$

Note that

$$\begin{aligned}\delta_n\delta'_n &= \delta_1\delta'_1 = (\lambda_2 - q^{-1}\lambda_1)(\lambda_2 - q\lambda_1) = \lambda_2^2 - q^{-1}\lambda_2\lambda_1 - q\lambda_1\lambda_2 + \lambda_1^2 \\ &= \lambda_2^2 - (q^{-1}+q)\lambda_2\lambda_1 + \lambda_1^2 \quad \text{for } n \geq 1.\end{aligned}$$

Now fix $b = -(q+q^{-1})\lambda_1\lambda_2 + \lambda_1^2 + \lambda_2^2$ and, for $\lambda \in \mathbb{F}$, let $p_\lambda(X)$ denote the quadratic

$$X^2 - (q+q^{-1})\lambda X + \lambda^2 - b.$$

Note that $p_{\lambda_2}(\lambda_1) = 0$. The sum of the roots of p_{λ_2} is $(q+q^{-1})\lambda_2$ and one root is λ_1 so the other is $(q+q^{-1})\lambda_2 - \lambda_1 = \lambda_3$. Similarly, if $j \in \mathbb{Z}$, the roots of P_{λ_j} are λ_{j-1} and λ_{j+1}.

Lemma 3.3.3. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be defined as in Lemma 3.3.1.*

(i) For $m > n$, $\lambda_n = \lambda_m$ if and only if $\lambda_{n-1} = \lambda_{m+1}$.

(ii) For $m > n > p$, we cannot have $\lambda_p = \lambda_n = \lambda_m$.

(iii) If the sequence $\{\lambda_i\}_{i\in\mathbb{Z}}$ has repetitions, then there exists $k \in \mathbb{Z}$ such that either $\lambda_k = \lambda_{k+1}, \lambda_{k-1} = \lambda_{k+2}, \lambda_{k-2} = \lambda_{k+3}, \ldots,$ with $\lambda_{k+1}, \lambda_{k+2}, \lambda_{k+3}, \ldots$ distinct, or $\lambda_{k-1} = \lambda_{k+1}, \lambda_{k-2} = \lambda_{k+2}, \lambda_{k-3} = \lambda_{k+3}, \ldots$ with $\lambda_k, \lambda_{k+1}, \lambda_{k+2}, \ldots$ distinct.

(iv) The set $\Lambda := \{\lambda_i : i \geq 1\}$ of values taken by λ_i is infinite.

Proof. (i) Let $m > n$ and suppose that $\lambda_n = \lambda_m$. Then $p_{\lambda_n} = p_{\lambda_m}$. So λ_{n-1} and λ_{n+1} are the roots of p_{λ_n} and the roots of p_{λ_m} are λ_{m-1} and λ_{m+1}. We cannot have $\lambda_{n-1} = \lambda_{m-1}$. To see this, suppose we have $\lambda_{n-1} = \lambda_{m-1}$. Then

$$A^{m-2}\begin{pmatrix}\lambda_2\\ \lambda_1\end{pmatrix} = \begin{pmatrix}\lambda_m\\ \lambda_{m-1}\end{pmatrix} = \begin{pmatrix}\lambda_n\\ \lambda_{n-1}\end{pmatrix} = A^{n-2}\begin{pmatrix}\lambda_2\\ \lambda_1\end{pmatrix}.$$

Then $\begin{pmatrix}\lambda_m\\ \lambda_{m-1}\end{pmatrix} = A^{m-n}\begin{pmatrix}\lambda_n\\ \lambda_{n-1}\end{pmatrix}$. This implies that $\begin{pmatrix}\lambda_n\\ \lambda_{n-1}\end{pmatrix}$ is an eigenvector for A^{m-n} with eigenvalue 1. But the eigenvalues of A^{m-n} are q^{m-n} and q^{n-m} and q is not a root of unity. So this cannot happen, whence $\lambda_{n-1} = \lambda_{m+1}$.

Conversely, by a similar method, if $\lambda_{n-1} = \lambda_{m+1}$ then $\lambda_n = \lambda_m$.

(ii) If $m > n > p$ and $\lambda_p = \lambda_n = \lambda_m$ then, by (i), $\lambda_{p-1} = \lambda_{n+1} = \lambda_{m-1}$ and $\begin{pmatrix} \lambda_p \\ \lambda_{p-1} \end{pmatrix}$ is an eigenvector for A^{m-p} with eigenvalue 1. This cannot happen, whence we cannot have $\lambda_p = \lambda_n = \lambda_m$.

(iii) It is clear from (i) and (ii).

(iv) It is immediate from (ii) or (iii). □

Lemma 3.3.4. *Let $\{\lambda_n\}_{n\geq 1}$ be as in Lemma 3.3.1. If the elements λ_i are not all distinct then there exists $n > 1$ such that*

(i) $\lambda_1 = \lambda_n$,

(ii) if $n = 2d$ is even then $\lambda_2 = \lambda_{n-1}, \ldots$ and $\lambda_d = \lambda_{d+1}$,

(iii) if $n = 2d+1$ is odd then $\lambda_2 = \lambda_{n-1}, \ldots$ and $\lambda_d = \lambda_{d+2}$,

(iv) the only equalities $\lambda_i = \lambda_j$, $i < j$, are those specified in (i), (ii) and (iii).

Proof. (i) This is clear from Lemma 3.3.3(i).

(ii) and (iii) are immediate from Lemma 3.3.3(iii).

(iv) This is immediate from Lemma 3.3.3(ii). □

We are now interested in a special case of the sequence considered above. Let $\eta \in \mathbb{F}\backslash\{0\}$. In Lemma 3.3.1, fix $\lambda_1 = \frac{q+\eta^2}{(1-q^2)\eta}$ and $\lambda_2 = \frac{1+q\eta^2}{(1-q^2)\eta}$, then $b = -q^{-1}$. Therefore $p_{\lambda_i}(x) = x^2 - (q+q^{-1})\lambda_i x + \lambda_i^2 + q^{-1}$. In this case we shall denote $\{\lambda_i\}_{i\geq 1}$ by $S(\eta)$. Thus, by Lemma 3.3.1, the sequence $S(\eta)$ can be written in terms of η, for $i \geq 1$,

$$\begin{aligned}
\lambda_i &= [i-1]\lambda_2 - [i-2]\lambda_1 \\
&= \frac{[i-1](1+q\eta^2)}{(1-q^2)\eta} - \frac{[i-2](q+\eta^2)}{(1-q^2)\eta} \\
&= \frac{q^{2-i}(1-q^{2i-2})(1+q\eta^2) - q^{3-i}(1-q^{2i-4})(q+\eta^2)}{(1-q^2)^2\eta} \\
&= \frac{q^{2-i}(1+q\eta^2 - q^{2i-2} - q^{2i-1}\eta^2 - q^2 - q\eta^2 + q^{2i-2} + q^{2i-3}\eta^2)}{(1-q^2)^2\eta} \\
&= \frac{q^{2-i}(1-q^2+q^{2i-3}\eta^2(1-q^2))}{(1-q^2)^2\eta}.
\end{aligned}$$

Thus

$$\lambda_i = q^{2-i}(1+q^{2i-3}\eta^2)/(1-q^2)\eta. \tag{3.10}$$

By Lemma 3.3.4, repetitions in $S(\eta)$ are determined by repetitions of λ_1 and by (3.10), for $n \geq 1$,

$$\begin{aligned}
\lambda_1 = \lambda_n \quad &\Leftrightarrow \quad \frac{q+\eta^2}{(1-q^2)\eta} = \frac{q^{2-n}(1+q^{2n-3}\eta^2)}{(1-q^2)\eta} \\
&\Leftrightarrow \quad q^{2-n} + q^{n-1}\eta^2 - (q+\eta^2) = 0 \\
&\Leftrightarrow \quad q^{2-n}(1-q^{n-1}) + \eta^2(q^{n-1}-1) = 0 \\
&\Leftrightarrow \quad (1-q^{n-1})(q^{2-n}-\eta^2) = 0.
\end{aligned}$$

Since q is not a root of unity,

$$\lambda_1 = \lambda_n \text{ if and only if } \quad \eta^2 = q^{2-n}. \tag{3.11}$$

Note that the alternative recurrence formulae (3.9) become

$$\begin{aligned}
\lambda_{n+1} &= q^{-1}\lambda_n + q^{n-1}\delta_1 = q^{-1}\lambda_n + q^{n-1}(\lambda_2 - q^{-1}\lambda_1) \\
&= q^{-1}\lambda_n + \frac{q^{n-1}(1+q\eta^2 - q^{-1}(q+\eta^2))}{(1-q^2)\eta} \\
&= q^{-1}\lambda_n - \frac{q^{n-2}(1-q^2)\eta^2}{(1-q^2)\eta} = q^{-1}\lambda_n - q^{n-2}\eta
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{n+1} &= q\lambda_n + q^{1-n}\delta_1' = q\lambda_n + q^{1-n}(\lambda_2 - q\lambda_1) \\
&= q\lambda_n + q^{1-n}\frac{1+q\eta^2 - q(q+\eta^2)}{(1-q^2)\eta} \\
&= q\lambda_n + \frac{q^{1-n}(1-q^2)}{(1-q^2)\eta} = q\lambda_n + q^{1-n}\eta^{-1}.
\end{aligned}$$

Thus

$$\lambda_{n+1} = q^{-1}(\lambda_n - q^{n-1}\eta) = q(\lambda_n + q^{-n}\eta^{-1}). \tag{3.12}$$

As a result of Lemma 3.3.4, there are three cases as follows.

Case I, all distinct : the terms of $S(\eta)$ are distinct. Thus $\eta^2 \neq q^{2-n}$ for all $n > 1$.

Case II, even repeating : $\eta^2 = q^{2-2d}$ for some positive integer d. In this case, we see that $\lambda_1 = \lambda_{2d}, \lambda_2 = \lambda_{2d-1}, \ldots, \lambda_d = \lambda_{d+1}$; otherwise the terms of $S(\eta)$ are distinct.

Case III, odd repeating : $\eta^2 = q^{2-(2d+1)}$ for some positive integer d. In this case, we see that $\lambda_1 = \lambda_{2d+1}, \lambda_2 = \lambda_{2d}, \ldots, \lambda_d = \lambda_{d+2}$; otherwise the terms of $S(\eta)$ are distinct.

3.4 Verma modules and finite-dimensional simple T_q-modules

Lemma 3.4.1. *Let $\eta, \lambda \in \mathbb{F}$ and let I be the left ideal $T_q(x-\lambda) + T_q(y - \eta z)$. Let J denote the ideal $T_q x + T_q y + T_q z$ (for which any two of the three given generators suffice), so that $\dim_{\mathbb{F}} T_q/J = 1$.*

(i) If $\lambda \neq 0$ and $\eta^2 + (q^2-1)\eta\lambda + q \neq 0$ then $I = T_q$.

(ii) If $\lambda = 0$ and $\eta^2 + q \neq 0$ then $I = J$.

(iii) If $\eta^2 - (1-q^2)\eta\lambda + q = 0$ then the linear map $\Gamma : f(z) \mapsto f(z) + I$ from $\mathbb{F}[z]$ to T_q/I is an isomorphism and $\dim_{\mathbb{F}} T_q/I = \infty$.

Proof. (i) and (ii). Let $\equiv$ denote equivalence modulo I. Observe that

$$\begin{aligned} x(y-\eta z) &= qyx + z - \eta q^{-1}(zx - y) \\ &\equiv q\lambda y + z - q^{-1}\eta\lambda z + \eta q^{-1} y \\ &\equiv q^{-1}(\eta^2 + (q^2-1)\eta\lambda + q)z. \end{aligned}$$

It follows that in (i), $z \in I$ and $y \equiv \eta z \in I$. As $x = yz - qzy \in I$ and $0 \neq \lambda \in I \cap \mathbb{F}$, it follows that $I = T_q$. In (ii), $\lambda = 0$ so $I \subseteq J$ and $x, z \in I$, therefore $J \subseteq I$.

(iii) Recall from the proof of Proposition 2.3.1 that T_q has a PBW-basis $\{z^i y^j x^k\}$ and a filtration in which $d(x) = d(y) = d(z) = 1$ and for which the associated graded ring is a coordinate ring of quantized 3-space with $xy = qyx$, $yz = qzy$ and $zx = q^{-1}xz$. As $q \neq 0$, $\eta \neq 0$ in (iii),

$$\begin{aligned}
& x(y-\eta z) = qyx + z - \eta q^{-1}(zx - y) \\
= \; & qy(x-\lambda) + q\lambda y + z - q^{-1}\eta z(x-\lambda) - q^{-1}\eta\lambda z + q^{-1}\eta(y-\eta z) + q^{-1}\eta^2 z \\
= \; & (qy - q^{-1}\eta z)(x-\lambda) + q^{-1}\eta(y-\eta z) + q\lambda(y-\eta z) \\
& +(q\eta\lambda + 1 - q^{-1}\eta\lambda + q^{-1}\eta^2)z \\
= \; & (qy - q^{-1}\eta z)(x-\lambda) + (q\lambda + q^{-1}\eta)(y-\eta z).
\end{aligned}$$

Thus $x(y-\eta z) \in \mathbb{F}(y-\eta z) + T_q(x-\lambda)$ and, if $x^{k-1}(y-\eta z) \in \mathbb{F}(y-\eta z) + T_q(x-\lambda)$ then $x^k(y-\eta z) \in \mathbb{F}x^{k-1}(y-\eta z) + x^{k-1}T_q(x-\lambda) \subseteq \mathbb{F}(y-\eta z) + T_q(x-\lambda)$ so it follows, inductively, that $x^k(y-\eta z) \in \mathbb{F}(y-\eta z) + T_q(x-\lambda)$ for all $k \geq 0$. Therefore $T_q(y-\eta z) \subseteq W(y-\eta z) + T_q(x-\lambda)$, where W is the subspace spanned by the monomials $z^i y^j$, $i, j \geq 0$. If $r = r_1(x-\lambda) \in T_q(x-\lambda)$ for some $r_1 \in T_q$ then, in $\mathrm{gr}(T_q)$, $\overline{r} \in \mathrm{gr}(T_q)\overline{x}$ so, if r has total degree d then the monomials $z^i y^j x^k$ of degree d with non-zero coefficients, when r is expressed in terms of the PBW basis, all satisfy $k > 0$. If $s = w(y-\eta z) \in W(y-\eta z)$ has degree e then, in $\mathrm{gr}(T_q)$, $\overline{s}$ has the form

$$(a_{e-1}\overline{y}^{e-1} + a_{e-2}\overline{z}\,\overline{y}^{e-2} + \cdots + a_0\overline{z}^{e-1})(\overline{y} - \eta\overline{z}), \quad a_i \in \mathbb{F}.$$

It follows that the monomials $z^i y^j x^k$ of degree e with non-zero coefficients, when s is expressed in terms of the PBW basis, all satisfy $k = 0$ and at least one satisfies $j > 0$. It follows that the leading term of $r + s$, which has degree $\max(d, e)$, must have a non-zero coefficient of a monomial $z^i y^j x^k$ with either $k > 0$ or $j > 0$. Hence if $0 \neq f(z) \in \mathbb{F}[z]$ then $f(z) \notin I$. Therefore Γ is injective. For $i, j, k \geq 0$, $z^i y^j x^k \equiv \lambda^k \eta^j z^{i+j} \mod I$ therefore $z^i y^j x^k \in \Gamma(\mathbb{F}[z])$. As T_q is spanned by the elements $z^i y^j x^k$, we see that Γ is surjective. Thus Γ is an isomorphism and $\dim_{\mathbb{F}}(T_q/I) = \infty$. $\square$

Remark 3.4.2. If $\eta = 0$ and $I = T_q(x - \lambda) + T_q(y - \eta z)$ then, by Lemma 3.4.1, $I = T_q$ if $\lambda \neq 0$ and $I = J$ if $\lambda = 0$. There is no value of λ for which Lemma 3.4.1(iii) is applicable. For $\eta \in \mathbb{F}\backslash\{0\}$, there is a unique value of λ, $\lambda = \frac{\eta^2+q}{(1-q^2)\eta}$, for which Lemma 3.4.1(iii) is applicable. For this value of λ we shall write I_η for $T_q(x - \lambda) + T_q(y - \eta z)$ and define the **Verma module** $V(\eta)$ to be the left T_q-module T_q/I_η.

Notation 3.4.3. By x**-eigenvector** (resp. x**-eigenvalue**) for a T_q-module M (or, more generally, a subspace W of M such that $xW \subseteq W$), we shall mean an eigenvector (resp. eigenvalue) for the action of x on M (or W). Use of such terms as x-eigenspace, generalized x-eigenvector and generalized x-eigenspace will be analogous. We shall see that $V(\eta)$ decomposes as a sum of generalized x-eigenspaces with x-eigenvalues determined by the sequence $S(\eta)$.

Lemma 3.4.4. *Let M be a left T_q-module and let $v \in M$ be an $x-$eigenvector, with $x-$eigenvalue λ. Let $W = \mathrm{Sp}(yv, zv)$, the subspace of M spanned by yv and zv and let $p_\lambda(X) = X^2 - (q + q^{-1})\lambda X + \lambda^2 + q^{-1}$. Then*

(i) $xW \subseteq W$;

(ii) if $W = 0$ then $\lambda = 0$ and $Jv = 0$, where J is the maximal ideal $T_qx+T_qy+T_qz$;

(iii) if $\dim_{\mathbb{F}} W = 1$ then $I_\eta v = 0$ for some $\eta \in \mathbb{F}\backslash\{0\}$ and zv and yv are x-eigenvectors, and their x-eigenvalue is a root of p_λ;

(iv) if $\dim_{\mathbb{F}} W = 2$ then the x-eigenvalues for W are the roots of $p_\lambda(X)$.

Proof. (i) We compute that

$$xyv = (qyx + z)v = q\lambda yv + zv \in W \quad \text{and} \tag{3.13}$$

$$xzv = q^{-1}(zx - y)v = q^{-1}\lambda zv - q^{-1}yv \in W. \tag{3.14}$$

Hence $xW \subseteq W$.

(ii) If $W = 0$ then $yv = zv = 0$ and also $xv = (yz - qzy)v = 0$. Thus $Jv = 0$. Also $\lambda = 0$ because $0 = xv = \lambda v$.

(iii) As $\dim_{\mathbb{F}} W = 1$, we see that yv and zv are linearly dependent and $\rho yv = \eta zv$ for some $\rho, \eta \in \mathbb{F}$, not both 0. If $\rho = 0$ then $zv = 0$ and $yv = (zx - qxz)v = zxv - qxzv = \lambda zv = 0$, whence $W = 0$, contradicting $\dim_{\mathbb{F}} W = 1$. Hence $\rho \neq 0$ and similarly $\eta \neq 0$. Without loss of generality we may assume that $\rho = 1$, in which case $xv = \lambda v$ and $yv = \eta zv$. Suppose that $\lambda \neq 0$. If $\eta^2 + (q^2 - 1)\eta\lambda + q \neq 0$ then by Lemma 3.4.1(i), $I_\eta = T_q$ and $T_q v = 0$ which is impossible. Hence, if $\lambda \neq 0$, then $\lambda = \frac{q+\eta^2}{(1-q^2)\eta}$. Now suppose that $\lambda = 0$. If $\eta^2 + q \neq 0$ then $I_\eta = J$ so $Jv = 0$ so $yv = zv = 0$ which is only true in case (ii). Therefore, if $\lambda = 0$, $\eta^2 + q = 0$ and, again $\lambda = \frac{q+\eta^2}{(1-q^2)\eta}$. Thus $I_\eta v = 0$.

In the sequence $S(\eta)$ from Lemma 3.3.4, $\lambda_1 = \lambda$ and, by (3.12), $q\lambda + \eta^{-1} = q^{-1}\lambda - q^{-1}\eta = \lambda_2$. Also λ_2 is a root of $p_\lambda(X)$. By (3.14), we see that

$$xzv = q^{-1}\lambda zv - q^{-1}yv = (q^{-1}\lambda - q^{-1}\eta)zv = \lambda_2 zv.$$

(iv) Here yv and zv form a basis for W. By (3.13) and (3.14), the action of x on W has trace $(q + q^{-1})\lambda$, determinant $\lambda^2 + q^{-1}$ and characteristic polynomial $X^2 - (q + q^{-1})\lambda X + \lambda^2 + q^{-1} = 0$. $\square$

3.5 Simple modules

Theorem 3.5.1. *Let M be a finite-dimensional simple left T_q-module. Then there exists $\eta \in \mathbb{F}\backslash\{0\}$ such that M is a homomorphic image of $V(\eta)$.*

Proof. Let $E = \{v \in M\backslash\{0\} : xv = \lambda v$ for some $\lambda \in \mathbb{F}\}$. Because $\dim_{\mathbb{F}} M < \infty$ and $\mathbb{F}$ is an algebraically closed field, the linear transformation $m_x : M \to M$, with $m_x(v) = xv$ for all $v \in M$, has an eigenvector so $E \neq \emptyset$. Suppose that for all $v \in E$, $\dim_{\mathbb{F}} \mathrm{Sp}(yv, zv) = 2$. Choose $v_1 \in E$, with $x-$eigenvalue λ_1 and let $W_1 = \mathrm{Sp}(yv_1, zv_1)$. By Lemma 3.4.4, $xW_1 \subseteq W_1$, with $x-$eigenvalues λ_0, λ_2, say. Let $v_2 \in W_1 \cap E$, with $x-$eigenvalue λ_2. By Lemma 3.4.4(iv),

$$\lambda_2^2 - (q + q^{-1})\lambda_1\lambda_2 + \lambda_1^2 + q^{-1} = 0,$$

from which we see that λ_1 and λ_2 cannot both be 0.

Let $W_2 = \mathrm{Sp}(yv_2, zv_2)$. By supposition, $\dim_{\mathbb{F}} W_2 = 2$ and, by Lemma 3.4.4, $xW_2 \subseteq W_2$ with $x-$eigenvalues being the roots of the equation

$$\lambda_2^2 - (q+q^{-1})\lambda_2 X + X^2 + q^{-1} = 0.$$

One of these roots must be λ_1. Denote the other, which could possibly be equal to λ_1, by λ_3 and let $v_3 \in W_2 \cap E$ with $x-$eigenvalue λ_3. By Lemma 3.4.4,

$$\lambda_3 + \lambda_1 = (q+q^{-1})\lambda_2.$$

Inductively, we produce a sequence $\{v_i : i \geq 1\}$ in E (not necessarily linearly independent), with a corresponding sequence $\{\lambda_i\}$ of $x-$eigenvalues, satisfying the following properties for $i \geq 1$:

- $W_i := \mathrm{Sp}(yv_i, zv_i)$ has dimension 2 and $v_{i+1} \in W_i$,
- $xW_i \subseteq W_i$ with $x-$eigenvalues λ_{i-1} and λ_{i+1};
- $p_\lambda(\lambda_{i+1}) = 0$ and $\lambda_{i+2} = (q+q^{-1})\lambda_{i+1} - \lambda_i$.

This sequence of $x-$eigenvalues is of the form considered in Lemma 3.3.1 and, by Lemma 3.3.3(iv), it takes infinitely many distinct values. This is impossible as $\dim_{\mathbb{F}} M < \infty$. So there must exist $v \in E$ for which $\dim_{\mathbb{F}} \mathrm{Sp}(yv, zv) < 2$. As M is simple, we have $M = T_q v$. If $\dim_{\mathbb{F}} \mathrm{Sp}(yv, zv) = 0$ then by Lemma 3.4.4(ii) $Jv = 0$ and $M \simeq T_q/J$, where J is the maximal ideal $T_q x + T_q y + T_q z$ of codimension 1. In this case, take $\eta^2 = -q$. Then $\lambda = 0$ and $I_\eta = T_q x + T_q(y - \eta z) \subseteq J$, whence $T_q/I_\eta \twoheadrightarrow T_q/J$. Thus there is a surjection $V(\eta) \twoheadrightarrow M$. If $\dim_{\mathbb{F}} \mathrm{Sp}(yv, zv) = 1$, by Lemma 3.4.4(iii) $I_\eta v = 0$ for some $\eta \in \mathbb{F}\backslash\{0\}$, whence there is a surjection $V(\eta) \twoheadrightarrow M$. $\square$

3.6 Basis for $V(\eta)$

In this section, we give some notation to define a basis for $V(\eta)$.

Lemma 3.6.1. *For $n \geq 0$, let $z_n = z^n + I_\eta \in V(\eta)$ and let*

$$V_{<n} = \mathrm{Sp}(\{z_m : 0 \leq m < n\})$$

(so $V_{<0} = 0$). Then

$$xz_n \equiv \lambda_{n+1} z_n \bmod V_{<n} \text{ and} \tag{3.15}$$

$$yz_n \equiv q^n \eta z_{n+1} \bmod V_{<n+1}, \tag{3.16}$$

where $S(\eta) = \{\lambda_i\}_{i \geq 0}$ is defined as in Section 3.3.

Proof. We proceed by induction on n. In the case when $n = 0$, (3.15) and (3.16) are true. Suppose they hold for some $n \geq 0$. Then

$$\begin{aligned} xz_{n+1} &= q^{-1}(zx - y)z_n \\ &\equiv (q^{-1}\lambda_{n+1} z(z^n + I_\eta) - q^{n-1}\eta z_{n+1}) \bmod V_{<n+1} \\ &\equiv q^{-1}(\lambda_{n+1} - q^{n-2}\eta) z_{n+1} \bmod V_{<n+1} \\ &\equiv \lambda_{n+2} z_{n+1} \bmod V_{<n+1} \quad \text{by (3.12)} \end{aligned}$$

and $yz_{n+1} = (qzy + x)z_n$. Here $xz_n \equiv \lambda_{n+1} z_{n+1} \bmod V_{<n}$ so $xz_n \equiv 0 \bmod V_{<n+1}$. Also $yz_n \equiv q^n \eta z_{n+1} \bmod V_{<n+1}$ so $qzyz_n \equiv q^{n+1}\eta z_{n+2} \bmod V_{<n+2}$. Therefore $yz_{n+1} \equiv q^{n+1}\eta z_{n+2} \bmod V_{<n+2}$. The result follows by induction. □

Remark 3.6.2. It is clear from Lemma 3.6.1 that, for $i \geq 1$, $xV_{<i} \subseteq V_{<i}$, $yV_{<i} \subseteq V_{<i+1}$ and $zV_{<i} \in V_{<i+1}$. Let $v_1 := 1 + I_\eta$ which is an $x-$eigenvector for λ_1.

Remark 3.6.3. The algebra T_q has a $\mathbb{Z}_2-$grading in which $\deg x = 0$ and $\deg y = \deg z = 1$. As the generators of I_η are homogeneous, the module $V(\eta)$ is a $\mathbb{Z}_2-$graded module. Thus $V(\eta)$ decomposes as the direct sum of its even part $V(\eta)_0$ and its odd part $V(\eta)_1$. Also $V(\eta)_0 = \bigoplus \mathbb{F} z_i$, i is even, and $V(\eta)_1$ is $\bigoplus \mathbb{F} z_i$, i is odd. As x is even, we have $xV(\eta)_0 \subseteq V(\eta)_0$ and $xV(\eta)_1 \subseteq V(\eta)_1$. For each $i \geq 1$, let $(V_{<i})_0 = V_{<i} \cap V(\eta)_0$ and $(V_{<i})_1 = V_{<i} \cap V(\eta)_1$. Thus $x(V_{<i})_0 \subseteq (V_{<i})_0$ and similarly, $x(V_{<i})_1 \subseteq (V_{<i})_1$.

Remarks 3.6.4. (i) Let $i \geq 1$. The $x-$eigenvalues for $V_{<i+1}$ are $\lambda_1, \lambda_2, \ldots, \lambda_{i+1}$. Because the sequence $\{\lambda_i\}$ takes no values three times, the generalized $x-$eigenspaces have degree ≤ 2. The terms λ_k, $1 \leq k \leq i+1$, where k is even, are the $x-$eigenvalues for $(V_{<i})_0$, and the other terms λ_k, where k is odd, are the $x-$eigenvalues for $(V_{<i})_1$.

Let $p = 0$ or 1 denote the parity of i, thus $z_i \in V(\eta)_p$ and by Lemma 3.6.1, $xz_i \equiv \lambda_{i+1} z_i \bmod (V_{<i})_p$. Therefore

$$xz_i = \lambda_{i+1} z_i + f_{i-1} \qquad \text{for some } f_{i-1} \in (V_{<i})_p.$$

If $\lambda_{i+1} \neq \lambda_k$ for any $k < i+1$, with the same parity as $i+1$, then $x - \lambda_{i+1}$ acts invertibly on $(V_{<i})_p$ so there exists $g_{i-1} \in (V_{<i})_p$ such that $f_{i-1} = (x - \lambda_{i+1}) g_{i-1}$. As $xz_i = \lambda_{i+1} z_i + (x - \lambda_{i+1}) g_{i-1}$, we have $x(z - g_{i-1}) = \lambda_{i+1}(z - g_{i-1})$, whence $v_{i+1} := z_i - g_{i-1} \in (V_{<i+1})_p$ is an x-eigenvector for λ_{i+1}.

If $\lambda_{i+1} = \lambda_k$ for some k, with the same parity as $i+1$, such that $k < i+1$, then by Lemma 3.3.3(ii), k is unique. We see that v_k spans the x-eigenspace for λ_{i+1} in $(V_{<i})_p$. There exist $g_{i-1} \in (V_{<i})_p$ and $\mu \in \mathbb{F}$ such that $f_{i-1} = (x - \lambda_{i+1}) g_{i-1} + \mu v_k$. Observe that

$$\begin{aligned}
(x - \lambda_{i+1})^2 (z_i - g_{i-1}) &= (x - \lambda_{i+1})(xz_i - \lambda_{i+1} z_i - (x - \lambda_{i+1}) g_{i-1}) \\
&= (x - \lambda_{i+1})(xz_i - \lambda_{i+1} z_i - f_{i-1} + \mu v_k) \\
&= \mu (x - \lambda_{i+1}) v_k = 0.
\end{aligned}$$

Now let $v_{i+1} := z_i - g_{i-1} \in (V_{<i+1})_p$. This is a generalized x-eigenvector in $\ker(x - \lambda_{i+1})^2$ and the generalized x-eigenspace for λ_{i+1} in $(V_{<i+1})_p$ is 2-dimensional, spanned by v_k and v_{i+1}.

(ii) The procedure outlined in (i) constructs a basis $\{v_1, v_2, v_3, \ldots\}$ for $V(\eta)$ such that the even part $V(\eta)_0$ has basis $\{v_1, v_3, v_5, \ldots\}$ and the set of elements $\{v_2, v_4, v_6, \ldots\}$ forms a basis for the odd part $V(\eta)_1$. We denote by $\mathcal{V}$ the basis $\{v_i\}_{i \geq 1}$ for $V(\eta)$.

Note that $v_2 = z + I_\eta$. Also $V(\eta)$ decomposes into a direct sum of generalized x-eigenspaces $E(\lambda_i) := \{v \in V(\eta) : (x - \lambda_i)^2 v = 0\}$, of dimension at most 2.

(iii) The behaviour of the sequence $S(\eta)$ of $x-$eigenvalues splits into three cases as mentioned in Section 3.4. In Case I, $\dim E(\lambda_i) = 1$ for each $i \geq 1$ and the action is diagonalisable. In Case II, $\dim E(\lambda_i) = 1$ if $i \geq 2d+1$ and $\dim E(\lambda_i) = 2$ if $1 \leq i \leq d$. In this case, whenever $\lambda_j = \lambda_k$ for some $k < j$ they have different parity. In this case, $E(\lambda_j)$ decomposes as the sum of two $1-$dimensional subspaces, $E(\lambda_j)_0$ and $E(\lambda_j)_1$, each spanned by an $x-$eigenvector with $x-$eigenvalue λ_j and the action of x on $V(\eta)$ is diagonalisable in Case II. There is some choice here: v_j may be replaced by $v_j + \rho v_i$ for any $\rho \in \mathbb{F}$.

In Case III, $\dim E(\lambda_i) = 1$ if $i = d+1$ or $i \geq 2d+2$ and $\dim E(\lambda_i) = 2$ if $1 \leq i \leq d$. In this case, whenever $\lambda_i = \lambda_j$ for some $i < j$ they have the same parity and v_j is an $x-$eigenvector with $x-$eigenvalue $\lambda_j = \lambda_i$. Thus $xv_j = \lambda_j v_j + \mu_j v_i$ for some $\mu_j \in \mathbb{F}$.

In all cases, we obtain $V(\eta) = \bigoplus_{i \in \mathcal{I}} E(\lambda_i)$, where

- $\mathcal{I} = \mathbb{N}$ in Case I,
- $\mathcal{I} = \mathbb{N} \backslash \{d+1, d+2, \ldots, 2d\}$ in Case II and
- $\mathcal{I} = \mathbb{N} \backslash \{d+2, d+3, \ldots, 2d+1\}$ in Case III.

For $0 \neq m \in V(\eta)$ and $i \in \mathcal{I}$, we say that i is in the **support of** m, $\mathrm{supp}(m)$, if the expression for m in $\bigoplus_{i \in \mathcal{I}} E(\lambda_i)$ has a non-zero summand from $E(\lambda_i)$. For a non-zero submodule N of $V(\eta)$, we define the **support of** N to be

$$\mathrm{supp}(N) = \bigcup_{0 \neq v \in N} \mathrm{supp}(v).$$

Lemma 3.6.5. *In all three cases, if N is a submodule of $V(\eta)$ and $i \in \mathrm{supp}(N)$ then $N \cap E(\lambda_i) \neq 0$.*

Proof. Let $0 \neq n \in N$. Then n can be written in the form $n = \sum_{j=1}^{m} n_j$ where $n_j \in E(\lambda_j)$. As $i \in \mathrm{supp}(N)$, choose $n \in N$ such that $n_i \neq 0$ and multiplying by $\prod_{k \neq i}(x - \lambda_k)^2$, $k \in \mathrm{supp}(n)$, we see that

$$\prod_{k \neq i}(x - \lambda_k)^2 n = \prod_{k \neq i}(x - \lambda_k)^2 n_i + \prod_{k \neq i}(x - \lambda_k)^2 \sum_{j \neq i}^{m} n_j.$$

Note that if $0 \neq v \in E(\lambda_j)$ then

$$(x-\lambda_k)^2 v \begin{cases} = 0, & j = k; \\ \neq 0, & j \neq k. \end{cases}$$

Thus $\prod_{k\neq i}(x-\lambda_k)^2 \sum_{j\neq i} n_j = 0$, but $\prod_{k\neq i}(x-\lambda_k)^2 n_i \neq 0$, whence

$$\prod_{k\neq i}(x-\lambda_j)^2 n = \prod_{k\neq i}(x-\lambda_k)^2 n_i \in (N \cap E(\lambda_i))\backslash\{0\}.$$

Thus $N \cap E(\lambda_i) \neq 0$. □

The next result covers Cases I and II.

Theorem 3.6.6. *Let $\eta \in \mathbb{F}\backslash\{0\}$ be such that $\eta^2 \neq q^{-m}$ for any odd positive integer m. The $\mathbb{F}$-basis $\mathcal{V}$ for $V(\eta)$ is such that, for $i \geq 1$,*

(i) $xv_i = \lambda_i v_i$,

(ii) $zv_i = v_{i+1} + \alpha_i v_{i-1}$,

(iii) $yv_i = q^{i-1}\eta v_{i+1} + \beta_i v_{i-1}$,

where, unless $i = 2$ and $\eta^2 = q$,

$$\alpha_i = \frac{q^{i-1}[i-1](1-q^{2(i-3)}\eta^4)}{(1-q^2)(1-q^{2i-5}\eta^2)(1-q^{2i-3}\eta^2)} \quad \textit{and} \tag{3.17}$$

$$\beta_i = q^{2-i}\eta^{-1}\alpha_i. \tag{3.18}$$

In the exceptional case, $\alpha_2 = 2q/((1-q^2)(1-q\eta^2))$ and $\beta_2 = \eta^{-1}\alpha_2$.

Proof. In Case I, for $i \geq 1$, $\dim E(\lambda_i) = 1$ and $E(\lambda_i)$ is spanned by an eigenvector v_i. By Remarks 3.6.4(i, ii), $v_i = z_i - g_{i-1} \in \mathbb{F}[z]$ for some $g_{i-1} \in V_{<i}$ and $\Gamma^{-1}(v_i)$ is monic of degree $i-1$. In particular, $v_1 = 1+I_\eta$ and, by Lemma 3.4.4(iii), $v_2 = zv_1 = z + I_\eta$. By Lemma 3.4.4(iii, iv), when $i > 1$, $zv_i, yv_i \in E(\lambda_{i-1}) + E(\lambda_{i+1})$. Thus $zv_i = v_{i+1}+\alpha_i v_{i-1}$, for some $\alpha_i \in \mathbb{F}$, and, by Lemma 3.6.1, $yv_i = q^{i-1}\eta v_{i+1}+\beta_i v_{i-1}$. Note that $\beta_1 = \alpha_1 = 0$.

In case II, where $\lambda_1 = \lambda_{2d}, \lambda_2 = \lambda_{2d-1}, \ldots, \lambda_d = \lambda_{d+1}$, $\dim E(\lambda_i) = 1$ for $i > 2d$ but $\dim E(\lambda_i) = 2$ for $1 \leq i \leq d$ when $E(\lambda_i)$ is spanned by two x-eigenvectors v_i and v_{2d+1-i} with different parity in the $\mathbb{Z}_2$-grading.

Let $p = 0$ or 1 be the parity of v_i. Then zv_i, $yv_i \in (E(\lambda_{i-1}) + E(\lambda_{i+1}))_{1-p} = \mathbb{F}v_{i-1} + \mathbb{F}v_{i+1}$. As in Case I, $zv_i = v_{i+1} + \alpha_i v_{i-1}$ for some $\alpha_i \in \mathbb{F}$, and similarly, $yv_i = q^{i-1}\eta v_{i+1} + \beta_i v_{i-1}$ for some $\beta_i \in \mathbb{F}$. Note that, in both cases, $\beta_1 = \alpha_1 = 0$ so (3.17) and (3.18) hold when $i = 1$. For $i > 1$, we compute that

$$\begin{aligned} yv_i &= (zx - qxz)v_i = \lambda_i(v_{i+1} + \alpha_i v_{i-1}) - qx(v_{i+1} + \alpha_i v_{i-1}) \\ &= \lambda_i v_{i+1} + \lambda_i \alpha_i v_{i-1} - q\lambda_{i+1} v_{i+1} - q\alpha_i \lambda_{i-1} v_{i-1} \\ &= (\lambda_i - q\lambda_{i+1})v_{i+1} + \alpha_i(\lambda_i - q\lambda_{i-1})v_{i-1} \\ &= q^{i-1}\eta v_{i+1} + \beta_i v_{i-1} \quad \text{by (3.12)} \end{aligned}$$

where $\beta_i = \alpha_i(\lambda_i - q\lambda_{i-1}) = \alpha_i q^{2-i}\eta^{-1}$ by (3.12). Then (3.17) holds. We now show that the formula (3.17) holds by induction on i. Let $j \geq 1$ and assume (3.17) holds when $i = j$. We have seen that (3.17) holds when $i = 1$. Observe that,

$$\begin{aligned} \lambda_j v_j = xv_j &= (yz - qzy)v_j = y(v_{j+1} + \alpha_j v_{j-1}) - qz(q^{j-1}\eta v_{j+1} + \beta_j v_{j-1}) \\ &= (q^j \eta v_{j+2} + \beta_{j+1} v_j) + \alpha_j(q^{j-2}\eta v_j + \beta_{j-1} v_{j-2}) \\ &\quad - q^j \eta(v_{j+2} + \alpha_{j+1} v_j) - q\beta_j(v_j + \alpha_{j-1} v_{j-2}) \\ &= (\beta_{j+1} + q^{j-2}\eta\alpha_j - q^j\eta\alpha_{j+1} - q\beta_j)v_j + (\alpha_j\beta_{j-1} - q\beta_j\alpha_{j-1})v_{j-2} \\ &= ((q^{1-j}\eta^{-1} - q^j\eta)\alpha_{j+1} + (q^{j-2}\eta - q^{3-j}\eta^{-1})\alpha_j)v_j \quad \text{by (3.18).} \end{aligned}$$

Hence

$$\lambda_j = (q^{1-j}\eta^{-1} - q^j\eta)\alpha_{j+1} + (q^{j-2}\eta - q^{3-j}\eta^{-1})\alpha_j. \tag{3.19}$$

Thus, by (3.10),

$$q^{1-j}\eta^{-1}(1 - q^{2j-1}\eta^2)\alpha_{j+1} = \lambda_j + q^{3-j}\eta^{-1}(1 - q^{2j-5}\eta^2)\alpha_j. \tag{3.20}$$

Note that when $j = 1$, so that $\alpha_1 = 0$, (3.20) gives the formula $\alpha_2 = \frac{q\eta(1+q^{-1}\eta^2)}{(1-q\eta^2)(1-q^2)\eta}$, so that (3.17) is valid when $i = 2$, including the exceptional case. Assume, for the moment, that $\eta^2 \neq q$. Multiplying throughout (3.20) by $q^{j-1}\eta$,

$$\begin{aligned}
(1-q^{2j-1}\eta^2)\alpha_{j+1} &= \frac{q\eta(1+q^{2j-3}\eta^2)}{(1-q^2)\eta} + q^2(1-q^{2j-5}\eta^2)\alpha_j \\
&= \frac{q(1+q^{2j-3}\eta^2)}{1-q^2} + \frac{q[j-1](1-q^{2j-5}\eta^2)(1-q^{2j-6}\eta^4)}{(1-q^2)(1-q^{2j-5}\eta^2)(1-q^{2j-3}\eta^2)} \\
&= \frac{q(1-q^2)(1-q^{4j-6}\eta^4) + q^3(1-q^{2j-2})(1-q^{2j-6}\eta^4)}{(1-q^2)^2(1-q^{2j-3}\eta^2)} \\
&= \frac{q-q^3-q^{4j-5}\eta^4+q^{4j-3}\eta^4+q^3-q^{2j-3}\eta^4-q^{2j+1}+q^{4j-5}\eta^4}{(1-q^2)^2(1-q^{2j-3}\eta^2)} \\
&= \frac{q(1-q^{2j})-q^{2j-3}\eta^4(1-q^{2j})}{(1-q^2)^2(1-q^{2j-3}\eta^2)}.
\end{aligned}$$

Thus

$$\alpha_{j+1} = \frac{q^j[j](1-q^{2j-3}\eta^4)}{(1-q^2)(1-q^{2j-1}\eta^2)(1-q^{2j-3}\eta^2)}.$$

In the case where $q = \eta^2$, where we have seen that (3.17) holds for $i = 2$, the above calculation is routinely amended, when $j = 2$, to avoid the appearance in denominators of $1 - q^{2j-5}\eta^2$ and remains valid when $j > 2$. This completes the inductive proof of (3.17). □

Lemma 3.6.7. *Let $\eta \in \mathbb{F}\backslash\{0\}$ be such that $\eta^2 \neq q^{-m}$ for any odd positive integer m and, for $j \geq 1$, let α_j and β_j be as in* (3.17) *and* (3.18).

(i) For $j \geq 1$, $\alpha_j = 0 \Leftrightarrow \beta_j = 0$.

(ii) There is at most value of $j > 1$ such that $\alpha_j = 0$.

(iii) If $j > 1$ and $\alpha_j = 0$ then $\sum_{k\geq j} \mathbb{F}v_k$ is a proper submodule of $V(\eta)$ of codimension $j-1$.

For each $j \geq 1$ there are four values of $\eta \in \mathbb{F}$ such that $\alpha_j = 0$.

Proof. (i) It is clear, from (3.18), that $\alpha_j = 0 \Leftrightarrow \beta_j = 0$.

(ii) This is immediate from (3.17) as q is not a root of unity.

(iii) This follows immediately from the actions of the generators specified in Theorem 3.6.6(i), (ii) and (iii).

When $j > 1$ and $\alpha_j = 0$, we have $1 - q^{2j-6}\eta^4 = 0$ which implies $\eta^4 = q^{2(3-j)}$. Thus, as $\operatorname{char}\mathbb{F} \neq 2$, there are four values of $\eta \in \mathbb{F}$ as required. $\square$

The next result covers Case I.

3.7 Case I: all distinct

Theorem 3.7.1. *Let $\eta \in \mathbb{F}\backslash\{0\}$ be such that $\eta^2 \neq q^{-m}$ for any non-negative integer m. For $i \geq 1$, let α_j and β_j be as in* (3.17) *and* (3.18).

(i) For $j > 1$, $\alpha_j = 0 \Leftrightarrow \eta^2 = -q^{3-j}$.

(ii) If $\alpha_j \neq 0$ for all $j > 1$ then $V(\eta)$ is simple.

(iii) If $j > 1$ and $\eta^2 = -q^{3-j}$ then $V(\eta)$ has a unique non-zero proper submodule $N(\eta) := \sum_{k\geq j} \mathbb{F}v_k$. Moreover $V(\eta)/N(\eta)$ is simple and has dimension $j - 1$.

Proof. (i) From (3.17), $\alpha_j = 0$ if and only if $j = 1$ or $q^{2(3-j)} - \eta^4 = 0$. Let $j > 1$. If $\alpha_j = 0$, then $q^{2(3-j)} - \eta^4 = 0$ and so $\eta^2 = \pm q^{3-j}$. If $j = 2$ then $\alpha_j \neq 0$ by the exceptional case of Theorem 3.6.6(iii) and $\eta^2 \neq q^{3-j}$. If $j \geq 3$ then $\eta^2 \neq q^{3-j}$ by the hypothesis $\eta^2 \neq q^{-m}$. Thus $\eta^2 = -q^{3-j}$.

Conversely, if $\eta^2 = -q^{3-j}$ then it is clear that $\alpha_j = 0$.

(ii) Suppose that $\alpha_j \neq 0$ for $j > 1$ and let N be a non-zero submodule of $V(\eta)$. Let $m = \min(\operatorname{supp}(N))$ and let $0 \neq v \in N$ with $m \in \operatorname{supp}(v)$. By Lemma 3.6.5, $v_m \in N$. If $m > 1$ then $\alpha_m \neq 0$ and $v_{m+1} + \alpha_m v_{m-1} = zv_m \in N$, whence $m - 1 \in \operatorname{supp}(N)$ which is a contradiction to the minimality of m. Thus $v_1 \in N$. As $V(\eta) = T_q v_1$, $N = V(\eta)$. Thus $V(\eta)$ is simple.

(iii) Suppose that $j > 1$ and that $\eta^2 = -q^{3-j}$. Thus $\alpha_n = 0$ if and only if $n = 1$ or $n = j$. By Lemma 3.6.7(iii), $N(\eta) := \sum_{k\geq j} \mathbb{F}v_k = T_q v_j$ is a proper submodule of $V(\eta)$ of codimension $j - 1$. Let N be a non-zero submodule of $V(\eta)$. The argument in (ii) shows that either $\min(\operatorname{supp}(N)) = j$, $v_j \in N$ and $N = N(\eta)$ or

$\min(\text{supp}(N)) = 1$, $v_1 \in N$ and $N = V(\eta)$. Thus $N(\eta)$ is the unique non-zero proper submodule of $V(\eta)$ and $V(\eta)/N(\eta)$ is simple. $\square$

Notation 3.7.2. Let $d \geq 1$ and $j = d+1$. Let $\eta \in \mathbb{F}$ be such that $\eta^2 = -q^{3-j}$. By (3.10),

$$\lambda_i = \frac{q^{2-i}(1+q^{2i-3}\eta^2)}{(1-q^2)\eta} = \frac{q^{2-i}(1-q^{2i-2d-1})}{(1-q^2)\eta}; \quad 1 \leq i \leq d.$$

It follows that $\lambda_i = -\lambda_{d+1-i}$, $1 \leq i \leq d$. Also, by (3.17) and (3.18),

$$\begin{aligned} \alpha_i &= -\frac{q^{2i-d-1}[i][d-i]}{(1-q^{2i-d-1})(1-q^{2i-d+1})}; \quad 1 \leq i \leq d-1; \\ \beta_i &= -\frac{q^{-i-d}[i][d-i]}{\eta(1-q^{2i-d-1})(1-q^{2i-d+1})}; \quad 1 \leq i \leq d-1. \end{aligned}$$

Let M_d denote the d-dimensional simple T_q-module $V(\eta)/N(\eta)$. We are abusing notation by writing v_n rather than $\overline{v_n}$ for the image in M_d of v_n. The actions of x, y, z are as follows

$$\begin{aligned} xv_1 = \lambda_1 v_1; \quad zv_1 &= v_2; \quad yv_1 = \eta v_2; \\ xv_i = \lambda_i v_i; \quad zv_i &= v_{i+1} + \alpha_i v_{i-1}; \quad yv_i = q^{i-1}\eta v_{i+1} + \beta_i v_{i-1}, \ (1 < i < d); \\ xv_d = \lambda_d v_d; \quad zv_d &= \alpha_d v_{d-1}; \quad yv_d = \beta_d v_{d-1}. \end{aligned}$$

With respect to the basis $\{v_i : 1 \leq i \leq d\}$ of M_d, the matrices representing multiplication by x, y, z respectively are:

$$\begin{pmatrix} \lambda_1 & 0 & & & & \\ 0 & \lambda_2 & 0 & & & \\ & 0 & \lambda_3 & & & \\ & & & \ddots & & \\ & & & & \lambda_{d-1} & 0 \\ & & & & 0 & \lambda_d \end{pmatrix}, \begin{pmatrix} 0 & \beta_1 & & & & \\ \eta & 0 & \beta_2 & & & \\ & q\eta & 0 & \beta_3 & & \\ & & & \ddots & & \\ & & & & 0 & \beta_{d-1} \\ & & & & q^{d-2}\eta & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \alpha_1 & & & & \\ 1 & 0 & \alpha_2 & & & \\ & 1 & 0 & \alpha_3 & & \\ & & & \ddots & & \\ & & & & 0 & \alpha_{d-1} \\ & & & & 1 & 0 \end{pmatrix}$$

It is clear that $\operatorname{tr} y = \operatorname{tr} z = 0$ and, as $\lambda_i = -\lambda_{d+1-i}$ that $\operatorname{tr} x = 0$. Note that if $d = 2t+1$ is odd then $\lambda_{t+1} = -\lambda_{t+1} = 0$.

There is, apparently, a second $d-$dimensional simple T_q-module $V(-\eta)/N(-\eta)$ and we shall establish that this module is isomorphic to M_d.

Theorem 3.7.3. *Let $d \geq 1$ and $j = d+1$. Let $\eta \in \mathbb{F}$ be such that $\eta^2 = -q^{3-j}$. The d-dimensional simple modules M_d and $V(-\eta)/N(-\eta)$ are isomorphic.*

Proof. We know from the sequence $\{\alpha_i\}_{i\geq 1}$ that

$$\alpha_i = \frac{q^{i-1}[i-1](1-q^{2(i-3)}\eta^4)}{(1-q^2)(1-q^{2i-5}\eta^2)(1-q^{2i-3}\eta^2)} \qquad \text{for } i \geq 1$$

that it is independent of the choice between η and $-\eta$. Secondly, let us consider the sequences $\{\lambda_i\}$ and $\{\beta_i\}$, for $i \geq 1$

$$\lambda_i = \frac{q^{2-i} - q^{2-(j-i)}}{(1-q^2)\eta} \qquad \text{and} \qquad \beta_i = q^{2-i}\eta^{-1}\alpha_i.$$

So the sequences $\{\lambda_i\}$ and $\{\beta_i\}$ are subject to multiplication by -1 when $-\eta$ replaces η.

Let $M = V(-\eta)/N(-\eta)$. Thus M has a basis $\{u_i : 1 \leq i \leq d\}$ such that

$$\begin{aligned} xu_1 = -\lambda_1 u_1; \quad zu_1 &= v_2; \quad yu_1 = -\eta v_2; \\ xu_i = -\lambda_i u_i; \quad zu_i &= u_{i+1} + \alpha_i u_{i-1}; \quad yu_i = -q^{i-1}\eta u_{i+1} - \beta_i u_{i-1},\ (1 < i < d); \\ xu_d = -\lambda_d u_d; \quad zu_d &= \alpha_d u_{d-1}; \quad yu_d = -\beta_d u_{d-1}. \end{aligned}$$

Hence, as $\lambda_d = -\lambda_1$ and $\beta_d = q^{2-d}\eta^{-1}\alpha_d = -\eta\alpha_d$,

$$\begin{aligned}(x-\lambda_1)u_d &= (x+\lambda_d)u_d = 0, \quad \text{and} \\ (y-\eta z)u_d &= -\beta_d u_{d-1} - \eta\alpha_d u_{d-1} = \alpha_d(q^{2-d}\eta^{-1}+\eta)u_d \\ &= \alpha_d\eta^{-1}(q^{3-j}+\eta^2)u_d = 0\end{aligned}$$

Thus $I_\eta \cdot u_d = 0$. As $M = T_q u_d$, by simplicity, there is a surjection $V(\eta) \twoheadrightarrow M$. As M_d is the unique simple factor of $V(\eta)$, $M \simeq M_d$. □

3.8 Case II: even repeating

Notation 3.8.1. Theorem 3.7.1 and 3.7.3 above provide one d-dimensional simple left T_q-module for each $d \geq 1$. We are now going to consider Case II, which will provide a further four $d-$dimensional simple modules for each d. In this case, let $\eta = q^{1-d}$ for some positive integer d and $\eta' = -\eta$. Thus $\eta^2 = \eta'^2 = q^{2-2d}$. In this situation, $\alpha_{2d+1} = 0$ and the sequence $S(\eta)$ begins as follows:

$$\lambda_1 = \lambda_{2d} = \frac{q^{1-d}+q^d}{1-q^2},\ \lambda_2 = \lambda_{2d-1} = \frac{q^{2-d}+q^{d-1}}{1-q^2}, \dots,\ \lambda_d = \lambda_{d+1} = \frac{1+q}{1-q^2}.$$

The sequence $\{\alpha_i\}$ has some symmetry as below: for $1 \leq j \leq 2d$, by (3.17),

$$\alpha_j = \frac{q^{2j-2d-3}[j-1][j-2d-1]}{(1-q^{2j-2d-3})(1-q^{2j-2d-1})}.$$

Using (3.8), we see that

$$\begin{aligned}\alpha_{2d-j+2} &= \frac{q^{2d-2j+1}[2d-j+1][1-j]}{(1-q^{2d-2j+1})(1-q^{2d-2j+3})} \\ &= \frac{q^{2d-2j+1}[j-2d-1][j-1]}{q^{4d-4j+4}(1-q^{2d-2j+1})(1-q^{2d-2j+3})} \\ &= \frac{q^{2j-2d-3}[j-2d-1][j-1]}{(1-q^{2j-2d-1})(1-q^{2j-2d-3})} = \alpha_j.\end{aligned}$$

Thus

$$\alpha_j = \alpha_{n-j+2}, \quad (1 \leq j \leq d). \tag{3.21}$$

The value of α_{d+1} does not occur in (3.21). By (3.8), we show that

$$\begin{aligned}
\alpha_{d+1} &= \frac{q^{-1}[d][-d]}{(1-q^{2(d+1)-2d-3})(1-q^{2(d+1)-2d-1})} \\
&= \frac{-q^{-1}([d])^2}{(1-q^{-1})(1-q)} \\
&= \frac{([d])^2}{(1-q)^2}.
\end{aligned}$$

Hence

$$\alpha_{d+1} = \left(\frac{[d]}{1-q}\right)^2. \tag{3.22}$$

As $\alpha_{2d+1} = 0$, by Lemma 3.6.7(iii), $V(\eta)$ has a $2d$-codimensional module $N(\eta) := \sum_{j\geq 2d+1} \mathbb{F}v_j$. Let $P(\eta)$ denote the $2d$-dimensional module $V(\eta)/N(\eta)$.

Lemma 3.8.2. *Let N be a submodule of $V(\eta)$. Let $1 \leq i \leq d$ and let $a, b \in \mathbb{F}$ be such that $av_i + bv_{2d+1-i} \in N \cap E(\lambda_i)$. Let $\epsilon = \frac{(1-q)}{[d]}$.*

(i) If $i > 1$ then $a\alpha_i v_{i-1} + bv_{2d+2-i} \in N \cap E(\lambda_{i-1})$ and if $i = 1$ then $bv_{2d+1} \in N \cap E(\lambda_{2d+1})$;

(ii) $av_{i+1} + b\alpha_{2d+1-i}v_{2d-i} \in N \cap E(\lambda_{i+1})$;

(iii) if $v_m \in N$ for some m with $1 \leq m \leq 2d$ then $N = V(\eta)$.

(iv) If $N \neq V(\eta)$, $i = d$ and $a = 1$ then $b = \pm\epsilon$.

Proof. (i) and (ii). For $i > 1$, by the action of z on $av_i + bv_{2d+1-i}$, we see that

$$\begin{aligned}
z(av_i + bv_{2d+1-i}) &= a(v_{i+1} + \alpha_i v_{i-1}) + b(v_{2d+2-i} + \alpha_{2d+1-i}v_{2d-i}) \\
&= av_{i+1} + a\alpha_i v_{i-1} + bv_{2d+2-i} + b\alpha_{2d+1-i}v_{2d-i}.
\end{aligned}$$

Here v_{i+1} and v_{2d-i} are $x-$eigenvectors with $x-$eigenvalue λ_{i+1} and v_{i-1} and v_{2d+2-i} are $x-$eigenvectors with $x-$eigenvalue λ_{i-1}. By the actions of $x - \lambda_{i+1}$ and $x - \lambda_{i-1}$, we have

$$a\alpha_i v_{i-1} + bv_{2d+2-i} \in N \cap E(\lambda_{i-1}) \tag{3.23}$$

$$av_{i+1} + b\alpha_{2d+1-i}v_{2d-i} \in N \cap E(\lambda_{i+1}). \tag{3.24}$$

If $i = 1$, by $z(av_1 + bv_{2d}) = av_2 + bv_{2d+1} + b\alpha_{2d}v_{2d-1}$, where v_2 and v_{2d-1} are $x-$eigenvectors with $x-$eigenvalue λ_2 and v_{2d+1} is an $x-$eigenvector with $x-$eigenvalue λ_{2d+1}. By the action of $x - \lambda_2$, $bv_{2d+1} \in N \cap E(\lambda_{2d+1})$.

(iii) If $m \leq d$, and apply (i) repeatedly to get $v_1 \in N$. If $m \geq d+1$ then, by successive applications of (ii), $v_d \in N$ and, using (i), $v_1 \in N$. As v_1 generates $V(\eta)$, we obtain $N = V(\eta)$.

(iv) Suppose that $N \neq V(\eta)$. By (iii), $\dim(N \cap E(\lambda_d)) < 2$, otherwise, $v_d \in N \cap E(\lambda_d)$. By (ii), $v_{d+1} + b\alpha_{d+1}v_d \in N \cap E(\lambda_d)$. As $\dim N \cap E(\lambda_d) < 2$, we see that $v_{d+1} + b\alpha_{d+1}v_d$ and $v_d + bv_{d+1}$ must be linearly dependent. Therefore $1 - b^2\alpha_{d+1} = 0$, whence $b^2 = (\alpha_{d+1})^{-1} = \epsilon^2$ by (3.22). Thus $b = \pm\epsilon$. $\square$

Lemma 3.8.3. *Let ϵ be as in Lemma 3.8.2 and, for $1 \leq j \leq d$, define $w_j, u_j \in N_j$ as follows:*

$$w_j = \epsilon^{-1}\rho_j v_j + v_{2d-j+1}, \quad u_j = -\epsilon^{-1}\rho_j v_j + v_{2d-j+1},$$

where $\rho_j = \prod_{\ell=j+1}^{d} \alpha_\ell$. (In particular, $w_d = \epsilon^{-1}v_d + v_{d+1}$ and $u_d = -\epsilon^{-1}v_d + v_{d+1}$.) Let $M_{1,d} = N(\eta) + \sum_{j=1}^{d} \mathbb{F}w_j$, $M_{2,d} = N(\eta) + \sum_{j=1}^{d} \mathbb{F}u_j$, $S_{1,d} = M_{1,d}/N(\eta) \subset P(\eta)$ and $S_{2,d} = M_{2,d}/N(\eta) \subset P(\eta)$. Then $M_{1,d}$ and $M_{2,d}$ are submodules of $V(\eta)$ and $S_{1,d}$ and $S_{2,d}$ are simple submodules of $P(\eta)$, of dimension d. Moreover $P(\eta) = S_{1,d} \oplus S_{2,d}$.

Proof. Here we compute the following actions of x, y, z on $w_1, \ldots, w_d$:

$$\begin{aligned}
xw_j &= \lambda_j w_j, \quad (1 \leq j \leq d), && (3.25)\\
zw_1 &= \alpha_2 w_2 + v_{2d+1}, && (3.26)\\
zw_j &= \alpha_{j+1}w_{j+1} + w_{j-1}, \quad (1 < j < d), && (3.27)\\
zw_d &= \epsilon^{-1}w_d + w_{d-1}, && (3.28)\\
yw_1 &= q^{1-d}\alpha_2 w_2 + q^d v_{2d+1}, && (3.29)\\
yw_j &= q^{j-d}\alpha_{j+1}w_{j+1} + q^{d-j+1}w_{j-1}, \quad (1 < j < d), && (3.30)\\
yw_d &= \epsilon^{-1}w_d + qw_{d-1}. && (3.31)
\end{aligned}$$

It follows that $M_{1,d}$ is a submodule of $V(\eta)$ and hence that $S_{1,d}$ is a submodule of $P(\eta)$. As $\dim(M_{1,d} \cap E(\lambda_j)) = 1$ for $1 \le j \le d$, $\dim(S_{1,d}) = d$. Let N be a submodule of $M_{1,d}$ strictly containing $N(\eta)$. By Lemma 3.6.5, $N \cap E(\lambda_j) \ne 0$ for some j, $1 \le j \le d$ and, by Lemma 3.8.2, $N \cap E(\lambda_j) \ne 0$ for all j, $1 \le j \le d$. As each $\dim(M_{1,d} \cap E(\lambda_j)) = 1$, $N = M_{1,d}$. Therefore $S_{i,d}$ is simple. Similarly, $S_{2,d}$ is a $d-$dimensional simple submodule of $P(\eta)$. By simplicity, $S_{1,d} \cap S_{2,d} = 0$, and by dimension, $P(\eta) = S_{1,d} \oplus S_{2,d}$. □

Theorem 3.8.4. *Let the notation be as in Lemma 3.8.3.*

(i) Let N be a non-zero proper submodule of $V(\eta)$. Then $N(\eta) = N$ or $N = M_{1,d}$ or $N = M_{2,d}$. The simple factors of $V(\eta)$ are $V(\eta)/M_{1,d} \simeq S_{2,d}$ and $V(\eta)/M_{2,d} \simeq S_{1,d}$. The module $V(-\eta)$ also has precisely two simple factors, $S_{3,d}$ and $S_{4,d}$, say. For $1 \le i \le 4$, $\dim S_{i,d} = d$.

(ii) No two of the four modules $S_{i,d}$ are isomorphic and none is isomorphic to the d-dimensional simple module M_d.

Proof. (i) Let $m = \min(\mathrm{supp}(N))$. By Lemma 3.6.5, $N \cap E(\lambda_m) \ne 0$. Note that $\alpha_i = 0$, if and only if $i = 1$ or $i = 2d+1$. By the action of z specified in Theorem 3.6.6 if $m > 2d$ then $m = 2d+1$ which implies $N = \sum_{j \ge m} \mathbb{F}v_j = N(\eta)$. We may now suppose that $m \le d$. (Recall that $m \in \mathcal{I} = \mathbb{N} \backslash \{d+1, \ldots, 2d\}$.)

By Lemma 3.8.2(i) and (ii), $N \cap E(\lambda_i) \ne 0$ for all i, $1 \le i \le d$. Thus $av_d + bv_{d+1} \in N$ for some a, $b \in \mathbb{F}$, not both 0. By Lemma 3.8.2(iii), if $v_{d+1} \in N$ then $N = V(\eta)$ so we may assume that $a = 1$. By Lemma 3.8.2(iv) either $w_d \in N$ or $u_d \in N$. By the action of z shown in the proof of Lemma 3.8.3, $N = M_{1,d}$ or $M_{2,d}$. Therefore the simple factor of $V(\eta)$ are $V(\eta)/M_{i,d}$, $i = 1, 2$. Now

$$V(\eta)/M_{1,d} \simeq P(\eta)/S_{1,d} \simeq (S_{1,d} \oplus S_{2,d})/S_{1,d} \simeq S_{2,d}.$$

Similarly, $V(\eta)/M_{2,d} \simeq S_{1,d}$. Replacing η by $-\eta$, we get the same result for $V(-\eta)$.

(ii) We use trace to distinguish the five simple d-dimensional modules M_d and $S_{i,d}$, $i = 1, 2, 3, 4$. We already know, from Notation 3.8.1, that, in M_d, $\operatorname{tr} x = \operatorname{tr} y =$

$\operatorname{tr} z = 0$. It is clear from (3.26) - (3.31) that, on $S_{1,d}$, y and z each act with trace ϵ^{-1}. Also

$$\begin{aligned}
\operatorname{tr} x = \sum_{j=1}^{d} \lambda_j &= \sum_{j=1}^{d} \frac{q^{j-d} + q^{d-j+1}}{1-q^2} \\
&= \frac{q^{1-d}(\sum_{k=0}^{2d-1} q^k)}{(1-q^2)} \\
&= \frac{q^{1-d}(\sum_{k=0}^{2d-1} q^k)(1-q)}{(1-q^2)(1-q)} \\
&= \frac{q^{1-d}(1-q^{2d})}{(1-q^2)(1-q)} \\
&= [d]/(1-q) = \epsilon^{-1}
\end{aligned}$$

Thus, in $S_{1,d}$, we have

$$\operatorname{tr} x = \operatorname{tr} y = \operatorname{tr} z = \frac{[d]}{1-q}.$$

In $S_{2,d}$ where $-\epsilon$ replaces ϵ, $\operatorname{tr} y = \operatorname{tr} z = -\epsilon^{-1}$, but $\operatorname{tr} x = \sum_{j=1}^{d} \lambda_j = \epsilon^{-1}$, as above.

Recall from Section 3.1 that T_q has an automorphism $\phi = \phi_y$ such that $\phi(y) = y$, $\phi(x) = -x$ and $\phi(z) = -z$. Then $\phi(I_\eta) = T_q(-x - \lambda_1) + T_q(y + \eta z) = I_{-\eta}$. It follows that $V(-\eta)$ is isomorphic to the module ${}^{\phi}V(\eta)$ where the action of T_q is defined by the rule $t \cdot v = \phi(t)v$.

Also the simple factor $S_{3,d}$ and $S_{4,d}$ must be isomorphic to ${}^{\phi}S_{1,d}$ and ${}^{\phi}S_{2,d}$. Renumber, if necessary, so that $S_{3,d} \simeq {}^{\phi}S_{1,d}$. Then the trace of x on $S_{3,d}$ is the trace of $\phi(x)$ on $S_{1,d}$, and similarly for y and z. As $\phi(y) = y$, but $\phi(x) = -x$ and $\phi(z) = -z$ on $S_{3,d}$,

$$\operatorname{tr} y = \epsilon^{-1}, \quad \operatorname{tr} x = \operatorname{tr} z = -\epsilon^{-1}$$

and in $S_{4,d}$,

$$\operatorname{tr} z = \epsilon^{-1}, \quad \operatorname{tr} x = \operatorname{tr} y = -\epsilon^{-1}.$$

Hence no two of the five modules M_d, $S_{i,d}$ can be isomorphic. □

Remark 3.8.5. When $d = 1$ the four simple modules $S_{i,d}$, $i = 1, 2, 3, 4$, are the four simple one-dimensional modules arising from the surjection from T_q to the group algebra $\mathbb{F}V$ as mentioned in Section 3.2.

3.9 Case III: odd repeating

Notation 3.9.1. We are now ready to consider Case III. In this case, we shall see that the modules $V(\eta)$ do not have a basis of $x-$eigenvalues but they are simple. Thus Case III provides no further finite-dimensional simple modules. We assume that η is one of the two solutions in $\mathbb{F}$ of $\eta^2 = q^{1-2d}$, and write $\eta = q^{\frac{1}{2}-d}$. Thus the repetitions in $S(\eta)$ are

$$\lambda_1 = \lambda_{2d+1},\ \lambda_2 = \lambda_{2d}, \ldots,\ \lambda_d = \lambda_{d+2}.$$

By (3.10),

$$\lambda_j = \frac{q^{2+d-j} + q^{j-d}}{(1-q^2)q^{1/2}}, \quad j \geq 1. \tag{3.32}$$

The sequences $\{\alpha_j\}$ and $\{\beta_j\}$ are defined as in Case I, but with gaps:

$$\alpha_j = \frac{-q^{2d-2j+2}[j-1][2d+2-j]}{(1-q^{2(d+2-j)})(1-q^{2(d+1-j)})}, \quad j \neq d+1, d+2 \tag{3.33}$$

$$\beta_j = q^{d-j+3/2}\alpha_j, \quad j \neq d+1, d+2. \tag{3.34}$$

Note that $\alpha_{2d+1} = \beta_{2d+1} = 0$, a potential obstruction to the simplicity of $V(\eta)$.

As in Case I,

$$xv_j = \lambda_j v_j; \quad 1 \leq j \leq d+1 \text{ or } j \geq 2d+2, \tag{3.35}$$

$$zv_j = v_{j+1} + \alpha_j v_{j-1}; \quad 1 \leq j \leq d \tag{3.36}$$

$$yv_j = q^{j-d-1/2}v_{j+1} + \beta_j v_{j-1}; \quad 1 \leq j \leq d. \tag{3.37}$$

For $d+2 \leq j \leq 2d+1$, there exists $\mu_j \in \mathbb{F}$ such that

$$xv_j = \lambda_j v_j + \mu_j v_{2d+2-j}. \tag{3.38}$$

Lemma 3.9.2. *The basis $\mathcal{V}$ can be chosen such that, for appropriate values of j and k, there exist $\alpha_j^{(k)}$, $\beta_j^{(k)} \in \mathbb{F}$ such that*

- $zv_{d+1} = v_{d+2}$,
  $yv_{d+1} = q^{\frac{1}{2}}v_{d+2} + \beta_{d+1}^{(1)}v_d$;

- $zv_{d+2} = v_{d+3} + \alpha^{(1)}_{d+2} v_{d+1}$,
  $yv_{d+2} = q^{\frac{3}{2}} v_{d+3} + \beta^{(1)}_{d+2} v_{d+1} + \beta^{(3)}_{d+2} v_{d-1}$;

- $zv_{d+r} = v_{d+r+1} + \alpha^{(1)}_{d+r} v_{d+r-1} + \alpha^{(2r-3)}_{d+r} v_{d-r+3}$,
  $yv_{d+r} = q^{\frac{2r-1}{2}} v_{d+r+1} + \beta^{(1)}_{d+r} v_{d+r-1} + \beta^{(2r-3)}_{d+r} v_{d-r+3} + \beta^{(2r-1)}_{d+r} v_{d-r+1}$,
  $(3 \le r \le d)$;

- $zv_{2d+1} = v_{2d+2} + \alpha^{(1)}_{2d+1} v_{2d} + \alpha^{(2d-1)}_{2d+1} v_2$,
  $yv_{2d+1} = q^{\frac{2d+1}{2}} v_{2d+2} + \beta^{(1)}_{2d+1} v_{2d} + \beta^{(2d-1)}_{2d+1} v_2$;

- $zv_{2d+2} = v_{2d+3} + \alpha^{(1)}_{2d+2} v_{2d+1} + \alpha^{(2d+1)}_{2d+2} v_1$,
  $yv_{2d+2} = q^{\frac{2d+3}{2}} v_{2d+3} + \beta^{(1)}_{2d+2} v_{2d+1} + \beta^{(2d+1)}_{2d+2} v_1$;

- $zv_{2d+r} = v_{2d+r+1} + \alpha^{(1)}_{2d+r} v_{2d+r-1}$,
  $yv_{2d+r} = q^{\frac{2d+2r-1}{2}} v_{2d+r+1} + \beta^{(1)}_{2d+r} v_{2d+r-1}$, $(r \ge 3)$.

Proof. Let $W := \mathrm{Sp}(yv_i, zv_i, yv_{2d+2-i}, zv_{2d+2-i})$. We find that

$$\begin{aligned}
xzv_i &= q^{-1}(zx-y)v_i = q^{-1}\lambda_i zv_i - q^{-1}yv_i,\\
xyv_i &= (qyx+z)v_i = q\lambda_i yv_i + zv_i,\\
xyv_{2d+2-i} &= (qyx+z)v_{2d+2-i} = q\lambda_i yv_{2d+2-i} + zv_{2d+2-i} + q\mu_i yv_i, \quad \text{and}\\
xzv_{2d+2-i} &= q^{-1}(zx-y)v_{2d+2-i} = q^{-1}\lambda_i zv_{2d+2-i} + q^{-1}\mu_i zv_i - q^{-1}yv_{2d+2-i}.
\end{aligned}$$

Thus W is invariant under the action of x. We shall show that $yv_{2d+2-i}, zv_{2d+2-i} \in E(\lambda_{i-1})+E(\lambda_{i+1})$. Let V be a finite-dimensional vector space with basis e_1, e_2, e_3, e_4 and let $\theta \in \mathrm{End}_{\mathbb{F}}(V)$ be the linear transformation such that

$$\begin{aligned}
\theta(e_1) &= q\lambda_i e_1 + e_2,\\
\theta(e_2) &= -q^{-1}e_1 + q^{-1}\lambda_i e_2,\\
\theta(e_3) &= q\mu_i e_1 + q\lambda_i e_3 + e_4 \quad \text{and}\\
\theta(e_4) &= q^{-1}\mu_i e_2 - q^{-1}e_3 + q^{-1}\lambda_i e_4.
\end{aligned}$$

Thus there is a surjective linear transformation $\phi : V \to S$ such that $\phi(e_1) = yv_i$, $\phi(e_2) = zv_i$, $\phi(e_3) = yv_{2d+2-i}$, $\phi(e_4) = zv_{2d+2-i}$ and $\phi(\theta(v)) = x\phi(v)$ for all

$v \in V$. The eigenvalues of θ on V are λ_{i-1} and λ_{i+1} (the root of p_{λ_i}) and each has multiplicity two. Therefore $V = V_1 \oplus V_2$ where V_1 is the generalized eigenspace for λ_{i-1} and V_2 is the generalized eigenspace for λ_{i+1}. Let $s_1 \in S$. Then there exist $v = v_1 + v_2$, $v_i \in V_i$, $i = 1, 2$, such that $s = \phi(v) = \phi(v_1) + \phi(v_2) = s_1 + s_2$, say. Thus

$$\begin{aligned}
(x - \lambda_{i-1})^2(s_1) &= (x - \lambda_{i-1})(xs_1 - \lambda_{i-1}s_1) \\
&= (x - \lambda_{i-1})(x\phi(v_1) - \lambda_{i-1}\phi(v_1)) \\
&= (x - \lambda_{i-1})(\phi\theta(v_1) - \lambda_{i-1}\phi(v_1)) \\
&= \phi\theta^2(v_1) - \lambda_{i-1}\phi\theta(v_1) - \lambda_{i-1}\theta\phi(v_1) + \lambda_{i-1}^2\phi(v_1) \\
&= \phi((\theta - \lambda_{i-1})^2(v_1)) = 0.
\end{aligned}$$

Thus $s_1 \in E(\lambda_{i-1})$. Similarly, $(x - \lambda_{i+1})^2(s_2) = 0$ and $s_2 \in E(\lambda_{i+1})$. Thus $S \subseteq E(\lambda_{i-1}) + E(\lambda_{i+1})$. In particular, zv_{2d+2-i}, $yv_{2d+2-i} \in E(\lambda_{i-1}) + E(\lambda_{i+1})$. Note that $\dim E(\lambda_i)$ has dimension at most 2 for $i \geq 1$ and $\dim E(\lambda_i) = 2$ if $d + 2 \leq i \leq 2d + 1$. We see that the space $E(\lambda_{i-1}) + E(\lambda_{i+1})$ has dimension 2 if $1 \leq d + 1$ or $i \geq 2d + 3$, dimension 3 if $i = d + 2$ or $2d + 1 \leq i \leq 2d + 2$ and has dimension 4 if $d + 3 \leq i \leq 2d$. By Lemma 3.6.1, zv_{2d+2-i} and yv_{2d+2-i} can be written as a linear combination of the appropriate basis elements in $V_{<2d+2-i}$. This is done, taking account of Lemma 3.6.1, in the display in the statement of the Lemma.

In the expression for zv_{d+r}, $1 \leq r \leq d$, there is no term $\alpha_{d+r}^{(2r-1)} v_{d-r+1}$. If we include such a term and then replace v_{d+r+1} by $v'_{d+r+1} := v_{d+r+1} + \alpha_{d+r}^{(2r-1)} v_{d-r+1}$ in (3.38), we see that

$$\begin{aligned}
xv'_{d+r+1} &= xv_{d+r+1} + \alpha_{d+r}^{(2r-1)} xv_{d-r+1} \\
&= \lambda_{d+r+1}v_{d+r+1} + \mu_{d+r+1}v_{d-r+1} + \alpha_{d+r}^{(2r-1)}\lambda_{d-r+1}v_{d-r+1} \\
&= \lambda_{d+r+1}v'_{d+r+1} + \mu_{d+r+1}v_{d-r+1} \qquad \text{as } \lambda_{d+r+1} = \lambda_{d-r+1}.
\end{aligned}$$

It follows that replacing v_{d+r+1} by v'_{d+r+1} does not affect (3.38). Thus we may set $\alpha_{d+r}^{(2r-1)} = 0$ as has been done in the statement of the Lemma. $\square$

Lemma 3.9.3. *In the Notation of 3.9.1 and Lemma 3.9.2, we have*

(i)

$$\begin{aligned}\mu_{d+2} &= -q^{-1}\beta^{(1)}_{d+1} = \frac{-q^{-\frac{1}{2}}[d][d+1]}{1-q^2} \neq 0, && (3.39)\\ \beta^{(1)}_{d+2} &= \mu_{d+2} + q^{-\frac{1}{2}}\alpha^{(1)}_{d+2}, && (3.40)\\ \beta^{(1)}_{d+r} &= q^{-r+3/2}\alpha^{(1)}_{d+r}, \quad (r \geq 3), && (3.41)\end{aligned}$$

(ii)

$$\alpha^{(1)}_{d+2} = \frac{q[2][d][d+1] + 2q}{(1-q^2)^2} \tag{3.42}$$

and, for $r \geq 3$, $\alpha^{(1)}_{d+r} = \alpha_{d+r}$ *and* $\beta^{(1)}_{d+r} = \beta_{d+r}$ *where* α_{d+r} *and* β_{d+r} *are as specified in* (3.33) *and* (3.34).

(iii) For $2 \leq r \leq d+1$,

$$\mu_{d+r} = -q^{-1}[r-1]\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\dots\alpha_{d-r+3} \neq 0. \tag{3.43}$$

Proof. (i) We use the formulae in Lemma 3.9.2 to express $yv_i = (zx - qxz)v_i$ and $xv_i = (yz - qzy)v_i$ for $i \geq 1$, as a linear combination of the basis $\mathcal{V}$. Note that, by (3.12), $\lambda_i - q\lambda_{i+1} = q^{j-d-\frac{1}{2}}$ and $\lambda_{j+1} - q\lambda_i = q^{d-j+\frac{1}{2}}$ for $i \geq 1$. Firstly, we compute that

$$\begin{aligned} yv_{d+1} &= (zx - qxz)v_{d+1} = \lambda_{d+1}zv_{d+1} - qxv_{d+2}\\ &= \lambda_{d+1}v_{d+2} - q(\lambda_{d+2}v_{d+2} + \mu_{d+2}v_d)\\ &= (\lambda_{d+1} - q\lambda_{d+2})v_{d+2} - q\mu_{d+2}v_d\\ &= q^{\frac{1}{2}}v_{d+2} - q\mu_{d+2}v_d.\end{aligned}$$

So in Lemma 3.9.2, $\beta^{(1)}_{d+1} = -q\mu_{d+2}$, and also $\mu_{d+2} = -q^{-1}\beta^{(1)}_{d+1}$. Secondly,

$$\begin{aligned}
yv_{d+2} &= (zx - qxz)v_{d+2} \\
&= z(\lambda_{d+2}v_{d+2} + \mu_{d+2}v_d) - qx(v_{d+3} + \alpha^{(1)}_{d+2}v_{d+1}) \\
&= \lambda_{d+2}(v_{d+3} + \alpha^{(1)}_{d+2}v_{d+1}) + \mu_{d+2}(v_{d+1} + \alpha_d v_{d-1}) \\
&\quad -q(\lambda_{d+3}v_{d+3} + \mu_{d+3}v_{d-1}) - q\alpha^{(1)}_{d+2}\lambda_{d+1}v_{d+1} \\
&= (\lambda_{d+2} - q\lambda_{d+3})v_{d+3} + (\alpha^{(1)}_{d+2}(\lambda_{d+2} - q\lambda_{d+1}) + \mu_{d+2})v_{d+1} \\
&\quad +(\mu_{d+2}\alpha_d - q\mu_{d+3})v_{d-1}.
\end{aligned}$$

As $\lambda_{d+2} - q\lambda_{d+3} = q^{3/2}$ and $\lambda_{d+2} - q\lambda_{d+1} = q^{-1/2}$, it follows that

$$yv_{d+2} = q^{3/2}v_{d+3} + \beta^{(1)}_{d+2}v_{d+1} + \beta^{(3)}_{d+2}v_{d-1}, \tag{3.44}$$

where

$$\beta^{(1)}_{d+2} = q^{-1/2}\alpha^{(1)}_{d+2} + \mu_{d+2} \quad \text{and} \quad \beta^{(3)}_{d+2} = \mu_{d+2}\alpha_d - q\mu_{d+3}. \tag{3.45}$$

For $r \geq 3$, we compute the equation $yv_{d+r} = (zx - qxz)v_{d+r}$ and compare the coefficients of v_{d+r-1}, v_{d-r+3} and also v_{d-r+1},

$$\begin{aligned}
yv_{d+r} &= (zx - qxz)v_{d+r} \\
&= z(\lambda_{d+r}v_{d+r} + \mu_{d+r}v_{d-r+2}) - qx(v_{d+r+1} + \alpha^{(1)}_{d+r}v_{d+r-1} + \alpha^{(2r-3)}_{d+r}v_{d-r+3}) \\
&= \lambda_{d+r}(v_{d+r+1} + \alpha^{(1)}_{d+r}v_{d+r-1} + \alpha^{(2r-3)}_{d+r}v_{d-r+3}) \\
&\quad +\mu_{d+r}(v_{d-r+3} + \alpha^{(1)}_{d-r+2}v_{d-r+1}) \\
&\quad -q(\lambda_{d+r+1}v_{d+r+1} + \mu_{d+r+1}v_{d-r+1}) \\
&\quad -q\alpha^{(1)}_{d+r}\lambda_{d+r-1}v_{d+r-1} - q\alpha^{(2r-3)}_{d+r}\lambda_{d-r+3}v_{d-r+3} \\
&= (\lambda_{d+r} - q\lambda_{d+r+1})v_{d+r+1} + (\lambda_{d+r} - q\lambda_{d+r-1})\alpha^{(1)}_{d+r}v_{d+r-1} \\
&\quad +((\lambda_{d+r} - q\lambda_{d-r+3})\alpha^{(2r-3)}_{d+r} + \mu_{d+r})v_{d-r+3} \\
&\quad +(\mu_{d+r}\alpha^{(1)}_{d-r+2} - q\mu_{d+r+1})v_{d-r+1}.
\end{aligned}$$

Hence

$$yv_{d+r} = q^{-r+\frac{1}{2}}v_{d+r-1} + \beta^{(1)}_{d+r}v_{d+r-1} + \beta^{(2r-3)}_{d+r}v_{d-r+3} + \beta^{(2r-1)}_{d+r}v_{d-r+1},$$

where

$$\begin{aligned}
\beta^{(1)}_{d+r} &= q^{-r+\frac{3}{2}}\alpha^{(1)}_{d+r}; \quad (r \geq 3) && (3.46)\\
\beta^{(2r-1)}_{d+r} &= \mu_{d+r}\alpha^{(1)}_{d-r+2} - q\mu_{d+r+1} && (3.47)\\
\beta^{(2r-3)}_{d+r} &= q^{-r+\frac{3}{2}}\alpha^{(2r-3)}_{d+r} + \mu_{d+r}. && (3.48)
\end{aligned}$$

It remains to show that $\mu_{d+2} = \frac{-q^{-\frac{1}{2}}[d][d+1]}{1-q^2} \neq 0$. To show this, we shall consider the equation $xv_{d+r-1} = (yz - qzy)v_{d+r-1}$ and compare the coefficients of v_{d+r-1} and v_{d-r+1}:

$$\begin{aligned}
xv_{d+r-1} &= (yz - qzy)v_{d+r-1}\\
&= y(v_{d+r} + \alpha^{(1)}_{d+r-1}v_{d+r-2} + \alpha^{(2r-5)}_{d+r-1}v_{d-r+4})\\
&\quad -qz(q^{\frac{2r-3}{2}}v_{d+r} + \beta^{(1)}_{d+r-1}v_{d+r-2} + \beta^{(2r-5)}_{d+r-1}v_{d-r+4} + \beta^{(2r-3)}_{d+r-1}v_{d-r+2})\\
&= q^{\frac{2r-1}{2}}v_{d+r+1} + \beta^{(1)}_{d+r}v_{d+r-1} + \beta^{(2r-3)}_{d+r}v_{d-r+3} + \beta^{(2r-1)}_{d+r}v_{d-r+1}\\
&\quad +\alpha^{(1)}_{d+r-1}yv_{d+r-2} + \alpha^{(2r-5)}_{d+r-1}yv_{d-r+4}\\
&\quad -q^{\frac{2r-1}{2}}(v_{d+r+1} + \alpha^{(1)}_{d+r}v_{d+r-1} + \alpha^{(2r-3)}_{d+r}v_{d-r+3})\\
&\quad -q\beta^{(1)}_{d+r-1}zv_{d+r-2} - q\beta^{(2r-5)}_{d+r-1}zv_{d-r+4} - q\beta^{(2r-3)}_{d+r-1}zv_{d-r+2}.
\end{aligned}$$

Here the following terms contribute to the coefficients of v_{d+r-1} or v_{d-r+1}:

$$\begin{aligned}
\alpha^{(1)}_{d+r-1}yv_{d+r-2} &= \alpha^{(1)}_{d+r-1}(q^{\frac{2r-5}{2}}v_{d+r-1} + \beta^{(1)}_{d+r-2}v_{d+r-3})\\
&\quad +\alpha^{(1)}_{d+r-1}(\beta^{(2r-7)}_{d+r-2}v_{d-r+5} + \beta^{(2r-5)}_{d+r-2}v_{d-r+3}),\\
q\beta^{(1)}_{d+r-1}zv_{d+r-2} &= q\beta^{(1)}_{d+r-1}(v_{d+r-1} + \alpha^{(1)}_{d+r-2}v_{d+r-3} + \alpha^{(2r-7)}_{d+r-2}v_{d-r+5})
\end{aligned}$$

and

$$q\beta^{(2r-3)}_{d+r-1}zv_{d-r+2} = q\beta^{(2r-3)}_{d+r-1}(v_{d-r+3} + \alpha^{(1)}_{d-r+2}v_{d-r+1}).$$

The coefficients of v_{d+r-1} in yv_{d-r+4} and zv_{d-r+4} are 0. Then

$$\begin{aligned}
xv_{d+r-1} &= (\beta^{(1)}_{d+r} + q^{\frac{2r-5}{2}}\alpha^{(1)}_{d+r-1} - q^{\frac{2r-1}{2}}\alpha^{(1)}_{d+r} - q\beta^{(1)}_{d+r-1})v_{d+r-1}\\
&\quad +(\beta^{(2r-3)}_{d+r} + \alpha^{(1)}_{d+r-1}\beta^{(2r-5)}_{d+r-2} - q^{\frac{2r-1}{2}}\alpha^{(2r-3)}_{d+r} - q\beta^{(1)}_{d+r-1}\alpha^{(1)}_{d+r-2})v_{d-r+3}\\
&\quad +(\beta^{(2r-1)}_{d+r-1} - q\beta^{(2r-3)}_{d+r-1}\alpha^{(1)}_{d-r+2})v_{d-r+1}.
\end{aligned}$$

As $xv_{d+r-1} = \lambda_{d+r-1}v_{d+r-1} + \mu_{d+r-1}v_{d-r+3}$, it follows that

$$\begin{aligned}
\lambda_{d+r-1} &= \beta^{(1)}_{d+r} + q^{\frac{2r-5}{2}}\alpha^{(1)}_{d+r-1} - q^{\frac{2r-1}{2}}\alpha^{(1)}_{d+r} - q\beta^{(1)}_{d+r-1}, && (3.49)\\
\text{and} \quad \beta^{(2r-1)}_{d+r-1} &= q\beta^{(2r-3)}_{d+r-1}\alpha^{(1)}_{d-r+2}. && (3.50)
\end{aligned}$$

We next look for $\beta^{(1)}_{d+1}$ for $d \geq 1$. From (3.49), when $r = 1$,

$$\begin{aligned}
\beta^{(1)}_{d+1} &= (q^{\frac{5}{2}} - q^{-\frac{3}{2}})\alpha_d + \lambda_d\\
&= \frac{q^{-3/2}(1-q^4)q^2[d-1][d+2]}{(1-q^2)(1-q^4)} + \frac{1+q^2}{(1-q^2)q^{\frac{1}{2}}}\\
&= \frac{q[d-1][d+2]}{(1-q^2)q^{\frac{1}{2}}} + \frac{1+q^2}{(1-q^2)q^{\frac{1}{2}}}
\end{aligned}$$

Using the equality in (3.8), we see that $[d-1][d+2] = \frac{q^{1-2d}(1-q^{2d-2})(1-q^{2d+4})}{(1-q^2)^2}$, and

$$\begin{aligned}
\beta^{(1)}_{d+1} &= \frac{q^{2-2d}(1-q^{2d-2}-q^{2d+4}+q^{4d+2}) + (1-q^2)(1-q^4)}{q^{\frac{1}{2}}(1-q^2)^3}\\
&= \frac{q^{2-2d} - q^{2d} + q^{2d+4} - q^4}{q^{\frac{1}{2}}(1-q^2)^3}\\
&= \frac{q^{2-2d}(1-q^{2d}) + q^4(q^{2d}-1)}{q^{\frac{1}{2}}(1-q^2)^3}\\
&= \frac{q^{2-2d}(1-q^{2d+2})(1-q^{2d})}{q^{\frac{1}{2}}(1-q^2)^3}.
\end{aligned}$$

Hence

$$\beta^{(1)}_{d+1} = \frac{q^{\frac{1}{2}}[d][d+1]}{1-q^2} \neq 0. \tag{3.51}$$

Therefore (i) is proved.

(ii) We next find $\alpha^{(1)}_{d+2}$. When $r = 2$, from (3.49), we obtain

$$\begin{aligned}
\lambda_{d+1} &= \beta^{(1)}_{d+2} - q^{\frac{3}{2}}\alpha^{(1)}_{d+2} - q\beta^{(1)}_{d+1}\\
&= q^{-\frac{1}{2}}\alpha^{(1)}_{d+2} + \mu_{d+2} - q^{\frac{3}{2}}\alpha^{(1)}_{d+2} - q\beta^{(1)}_{d+2} \quad \text{by (3.45)}.
\end{aligned}$$

By using (3.51), we see that

$$\begin{aligned}\alpha^{(1)}_{d+2} &= \frac{q^{\frac{1}{2}}}{1-q^2}(q^{-1}(1+q^2)\beta^{(1)}_{d+1}+\lambda_{d+1})\\ &= \frac{q^{\frac{1}{2}}}{1-q^2}\left(\frac{q^{-1}(1+q^2)q^{1/2}[d][d+1]}{1-q^2}+\frac{2q^{1/2}}{1-q^2}\right)\\ &= \frac{q[2][d][d+1]+2q}{(1-q^2)^2}.\end{aligned}$$

It remains to show that, for $r\geq 3$, $\alpha^{(1)}_{d+r}=\alpha_{d+r}$ and $\beta^{(1)}_{d+r}=\beta_{d+r}$.

If $r>3$, substituting $\beta^{(1)}_{d+r}$ and $\beta^{(1)}_{d+r-1}$ by $q^{-r+\frac{3}{2}}\alpha^{(1)}_{d+r}$ and $q^{-r+\frac{5}{2}}\alpha^{(1)}_{d+r-1}$ respectively, (3.49) becomes

$$\begin{aligned}\lambda_{d+r-1} &= q^{-r+\frac{3}{2}}\alpha^{(1)}_{d+r}-q^{r+\frac{1}{2}}\alpha^{(1)}_{d+r}-q^{-r+\frac{7}{2}}\alpha^{(1)}_{d+r-1}+q^{-r+\frac{5}{2}}\alpha^{(1)}_{d+r-1}\\ &= (q^{-r+\frac{3}{2}}-q^{r+\frac{1}{2}})\alpha^{(1)}_{d+r}+(q^{r-\frac{5}{2}}-q^{-r+\frac{7}{2}})\alpha^{(1)}_{d+r-1}.\end{aligned}\tag{3.52}$$

Setting $\eta=q^{-d+1/2}$ and $i\geq d+3$ in the equation (3.19), we see that

$$\begin{aligned}\lambda_{d+r-1} &= \alpha_{d+r}(q^{1-(d+r-1)}q^{d-1/2}-q^{d+r}q^{-d+1/2})\\ &\quad +\alpha_{d+r-1}(q^{d+r-3}q^{-d+1/2}-q^{3-(d+r-1)}q^{d-1/2}).\end{aligned}\tag{3.53}$$

The matching formulae (3.52) and (3.53) express $\alpha^{(1)}_{d+r}$, respectively α_{d+r}, in terms of $\alpha^{(1)}_{d+r-1}$ and λ_{d+r-1}, respectively α_{d+r-1} and λ_{d+r-1}. To show that $\alpha^{(1)}_{d+r}=\alpha_{d+r}$ when $r\geq 3$ it is therefore enough to show that $\alpha^{(1)}_{d+3}=\alpha_{d+3}$.

When $r=3$, using (3.45) and (3.46) to substitute for $\beta^{(1)}_{d+2}$ and $\beta^{(1)}_{d+3}$, in (3.49), we get

$$\begin{aligned}\lambda_{d+2} &= \beta^{(1)}_{d+3}+q^{\frac{1}{2}}\alpha^{(1)}_{d+2}-q^{\frac{5}{2}}\alpha^{(1)}_{d+3}-q\beta^{(1)}_{d+2}\\ &= (q^{-\frac{3}{2}}-q^{\frac{5}{2}})\alpha^{(1)}_{d+3}-q(\mu_{d+2}+q^{\frac{1}{2}}\alpha^{(1)}_{d+3})+q^{\frac{1}{2}}\alpha^{(1)}_{d+2}\\ &= (q^{-\frac{3}{2}}-q^{\frac{5}{2}})\alpha^{(1)}_{d+3}-q\mu_{d+2}.\end{aligned}$$

Hence

$$\begin{aligned}
\alpha^{(1)}_{d+3} &= \frac{q^{3/2}}{1-q^4}(\lambda_{d+2}+q\mu_{d+2})\\
&= \frac{q^{3/2}}{1-q^4}\left(\frac{1+q^2}{q^{1/2}(1-q^2)}-\frac{q[d][d+1]}{q^{1/2}(1-q^2)}\right)\\
&= \frac{q^{1-2d}}{(1-q^4)(1-q^2)}\left(\frac{q^{2d}(1-q^2)(1-q^4)-q^2(1-q^{2d})(1-q^{2d+2})}{(1-q^2)^2}\right)\\
&= -q^2\frac{[d-1][d+2]}{(1-q^2)(1-q^4)}=\alpha_{d+3}.
\end{aligned}$$

Therefore $\alpha^{(1)}_{d+r}=\alpha_{d+r}$ for $r\geq 3$.

(iii) Let $2\leq r\leq d$. We proceed by induction on r. By an easy induction, it follows from (3.50) that $\beta^{(2k-1)}_{d+k}=q^{k-1}\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\cdots\alpha_{d-k+3}$ for $2\leq k\leq r$. We now prove (3.43) by induction on r. In the case when $r=2$, we see that $\mu_{d+2}=-q^{-1}\beta^{(1)}_{d+1}$, so (3.43) holds. Assume that $r>2$ and that (3.43) holds for $2\leq k\leq r$, that is,

$$\mu_{d+k}=-q^{-1}[k-1]\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\cdots\alpha_{d-k+3}$$

Observe that, by (3.47),

$$\begin{aligned}
\mu_{d+r+1} &= q^{-1}(\mu_{d+r}\alpha_{d-r+2}-\beta^{(2r-1)}_{d+r})\\
&= q^{-1}(-q^{-1}[r-1]\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\cdots\alpha_{d-r+2}-q^{r-1}\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\cdots\alpha_{d-r+2})\\
&= -q^{-2}([r-1]+q^r)\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\ldots\alpha_{d-r+2}\\
&= \frac{-q^{-2}(q^{2-r}(1-q^{2r-2})+q^r-q^{r+2})}{1-q^2}\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\ldots\alpha_{d-r+2}\\
&= -q^{-1}[r]\beta^{(1)}_{d+1}\alpha_d\alpha_{d-1}\ldots\alpha_{d-r+2}.
\end{aligned}$$

Hence (iii) is proved. □

Proposition 3.9.4. *If $V(\eta)$ is not simple then it has a unique proper non-zero submodule N. Moreover, if N exists, $\dim V(\eta)/N=2d+1$ and x acts on N with non-zero trace whereas z and y act with trace 0.*

Proof. Suppose that $V(\eta)$ is not simple and let N be a non-zero submodule of $V(\eta)$. Let $i=\min(\operatorname{supp}N)$. Thus $1\leq i\leq d+1$ or $i\geq 2d+2$. We know that $\dim E(\lambda_i) = 2$ if $1 \le i \le d$ and $\dim E(\lambda_i) = 1$ if $i = d+1$ or $i \ge 2d+2$. By Lemma 3.6.5, $N \cap E(\lambda_i) \ne 0$. We claim that $v_i \in N$. This is clear when $\dim E(\lambda_i) = 1$. Suppose that $1 \le i \le d$. There exist $a, b \in \mathbb{F}$, not both 0, such that $t = av_i + bv_{2d+2-i} \in N$. If $b = 0$ then $v_i \in N$, and if $b \ne 0$ then

$$\begin{aligned}(x-\lambda_i)t &= a(x-\lambda_i)v_i + b(x-\lambda_i)v_{2d+2-i} \\ &= b(\lambda_{2d+2-i}v_{2d+2-i} + \mu_{2d+2-i}v_{2d+2-i} - \lambda_i v_{2d+2-i})\end{aligned}$$

Recall that $\lambda_i = \lambda_{2d+2-i}$, $(x-\lambda_i)t = b\mu_{2d+2-i}v_{2d+2-i} \in N$. By Lemma 3.9.3, $\mu_{2d+2-i} \ne 0$, it follows that $v_i \in N$. Thus $v_i \in N$ in all cases. Since $N \ne V(\eta)$, we know that $i \ne 1$. By the action of z on v_i, we see that $zv_i = v_{i+1} + \alpha_i v_{i-1} \in N$. If $\alpha_i \ne 0$ then $i-1 \in \operatorname{supp}(N)$, contradicting the minimality of i, so $i = 2d+2$. It follows, from the minimality of i, that $\alpha_{2d+2}^{(2d+1)} = 0 = \beta_{2d+2}^{(2d+1)}$, whence $\sum_{i \ge 2d+2} \mathbb{F}v_i$ is a submodule of $V(\eta)$ of codimension $2d+1$. As $v_{2d+2} \in N$, using the action of z, we see that $N = \sum_{i \ge 2d+2} \mathbb{F}v_i$. Therefore N is the unique non-zero proper submodule of $V(\eta)$. It follows that $L = V(\eta)/N(\eta)$ is the unique simple image of $V(\eta)$. The trace of x acting on L is

$$\begin{aligned}\operatorname{tr} x = \sum_{j=1}^{2d+1} \lambda_j &= \sum_{j=1}^{2d+1} \frac{(q^{2+d-j} + q^{j-d})}{(1-q^2)q^{1/2}} \\ &= \frac{2q^{-d}(\sum_{k=0}^{2d+1} q^k)}{q^{1/2}(1-q^2)} \\ &= \frac{2q^{\frac{1}{2}-d}[2d+1]_q}{(1-q^2)} \ne 0,\end{aligned}$$

where $[n]_q = \frac{q^n - 1}{q-1}$ for any integer n. It is clear from the coefficient of yv_i and zv_i that $\operatorname{tr} y = 0 = \operatorname{tr} z$. $\square$

Theorem 3.9.5. *The modules $V(\eta)$ and $V(-\eta)$ are simple.*

Proof. Suppose that $V(\eta)$ is not simple and let L be the unique simple factor of $V(\eta)$. Note that $V(-\eta) \simeq {}^{\phi_y}V(\eta)$, by the proof of Theorem 3.8.4, where $\phi = \phi_y$. By Proposition 3.9.4, $V(-\eta)$ has a unique simple factor L', say. On L', x acts with non-zero trace while $\operatorname{tr} y = \operatorname{tr} z = 0$. Using Proposition 3.9.4 again, the simple modules L and L' are, up to isomorphism, the only $(2d+1)-$dimensional simple modules arising in Case III.

Note that, by Notation 3.7.2, Theorem 3.8.4 and Proposition 3.9.4, every finite-dimensional simple T_q-module has either

(i) $\operatorname{tr} x = \operatorname{tr} y = \operatorname{tr} z = 0$ in Case I;

(ii) $\operatorname{tr} x \neq 0,\ \operatorname{tr} y \neq 0,\ \operatorname{tr} z \neq 0$ in Case II or

(iii) $\operatorname{tr} x \neq 0,\ \operatorname{tr} y = \operatorname{tr} z = 0$ in Case III.

Consider the simple module ${}^{\sigma}L$ where σ is the automorphism such that $\sigma(x) = y$, $\sigma(y) = z$ and $\sigma(z) = x$. On this module, $\operatorname{tr} x = \operatorname{tr} y = 0$ but $\operatorname{tr} z \neq 0$. None of the modules in (i)-(iii) satisfy $\operatorname{tr} x = 0$ and $\operatorname{tr} z \neq 0$. This is a contradiction and therefore $V(\eta)$ and $V(-\eta)$ are simple. $\square$

Theorem 3.9.6. *For $d \geq 1$, T_q-module has, up to isomorphism, exactly five $d-$dimensional simple modules M_d, $S_{1,d}$, $S_{2,d}$, $S_{3,d}$ and $S_{1,d}$.*

Proof. This is immediate from Theorem 3.7.1, Theorem 3.7.3, Lemma 3.8.3 and Theorem 3.8.4. $\square$

3.10 Leonard Triples

Let $d \geq 1$. For each of the five d-dimensional simple T_q-modules M, the matrix representing x with respect to the presented basis of M is diagonal while those representing y and z are irreducible tridiagonal.

Consider the $d-$dimensional simple module ${}^{\sigma}M$, where σ is the $\mathbb{F}-$automorphism in Section 3.1. Then ${}^{\sigma}M$ must have a basis as described above. Therefore M has a basis for which the action of y is diagonal while those representing x and z are irreducible tridiagonal. The same method, with ${}^{\sigma^2}M$ replacing σ, shows that M

has the third basis for which the matrix representing z is diagonal and those representing x and y are irreducible tridiagonal. Thus each of the five d-dimensional simple T_q-modules gives rise to a Leonard triple.

Chapter 4

Poisson modules

In this chapter, we study a Poisson algebra related to T_q and classify the finite-dimensional simple Poisson modules for this Poisson algebra. We find that there are five $d-$dimensional simple Poisson modules for each $d \geq 1$. This corresponds to Theorem 3.9.6. We also study a Poisson algebra arising from the quantized enveloping algebra $U_q(sl_2)$ using a presentation discovered by Ito, Terwilliger and Weng [23] and prove a simple module for this Poisson algebra.

4.1 Poisson algebras

Let A be a Poisson algebra. We need to distinguish between two possible meanings of the term maximal Poisson ideal. By **maximal Poisson ideal**, we shall mean a Poisson ideal I of A such that if J is a Poisson ideal and $I \nsubseteq J$ then $J = A$. By **Poisson maximal ideal**, we shall mean a maximal ideal of A that is also a Poisson ideal. For example, let $A = \mathbb{C}[x, y]$ which is a Poisson algebra with the Poisson bracket $\{x, y\} = 1$. Then 0 is a maximal Poisson ideal but is not a Poisson maximal ideal.

We define a Poisson module from Definition 1.9.8 and give important information on annihilators of simple Poisson modules.

Lemma 4.1.1. *Let A be a Poisson $\mathbb{C}$-algebra and M a Poisson $A-$module.*

(i) The annihilator of M is a Poisson ideal of A;

(ii) if M is a simple Poisson module then the annihilator of M is a prime Poisson ideal of A;

(iii) if M is a finite-dimensional simple Poisson module then the annihilator of M is a Poisson maximal ideal of A.

Proof. (i). Let $a'M = 0$ for some $a' \in A$. Then for all $a \in A$ and $m \in M$,

$$0 = \{a, a'm\}_M = \{a, a'\}m + a'\{a, m\}_M = \{a, a'\}m,$$

so $\{a, a'\} \in \operatorname{ann}_A M$. It follows that $\operatorname{ann}_A M$ is a Poisson ideal of A.

(ii). Let M be a simple Poisson module, let I and J be Poisson ideals of A. Let $P = \operatorname{ann}_A(M)$. Suppose that $IJ \subseteq P$, that is, $IJM = 0$. We want to show that $I \subseteq P$ or $J \subseteq P$. We show that JM is a Poisson submodule of M. Let $j \in J$ and $m \in M$. For $a \in A$,

$$\{a, jm\}_M = j\{a, m\}_M + \{a, j\}m \in JM.$$

Hence JM is a Poisson submodule of M. Since M is a simple Poisson module, $JM = 0$ or $JM = M$. If $JM = M$ then $IM = IJM = 0$, so $J \subseteq P$ or $I \subseteq P$. This implies P is a Poisson-prime ideal of M, and also a prime Poisson ideal of M by Theorem 1.9.6.

(iii) Let M be a finite-dimensional simple Poisson module and $P = \operatorname{ann}_A(M)$. By (ii), P is a prime Poisson ideal of A. Then M is a faithful A/P-module. Let θ be the map from A/P to the endomorphism ring of M, $\operatorname{End}_{\mathbb{C}}(M)$, given by $\theta(a + P)(m) = am$ for $a \in A$ and $m \in M$. We claim that θ is an injective $\mathbb{C}$-homomorphism. For $\overline{a}$, $\overline{b} \in A/P$, if $\overline{a} = \overline{b}$ then $a - b \in P$ and $0 = (a - b)m$. Thus $\theta(\overline{a})(m) = am = bm = \theta(\overline{b})(m)$, and θ is well-defined.

Let a, $b \in A$. We now look at $\theta(\overline{a}\ \overline{b})(m) = abm$ and $\theta(\overline{a})\theta(\overline{b})(m) = \theta(\overline{a})bm = abm$. Therefore $\theta(\overline{a}\ \overline{b}) = \theta(\overline{a})\theta(\overline{b})$, whence θ is a $\mathbb{C}-$homomorphism. Next, we show that θ is injective. Let $\overline{a} \in A/P$ be such that $\theta(\overline{a}) = 0$. Then $am = 0$

for all $m \in M$ and $a \in P$ which implies $\ker\theta = 0$. Hence θ is injective. As $\dim_{\mathbb{C}}(M) < \infty$, $\dim_{\mathbb{C}}(A/P) \leq \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M) = (\dim_{\mathbb{C}}(M))^2$. Since A/P is a finite-dimensional algebra over $\mathbb{C}$, it follows that A/P is an Artinian ring. We also know that A/P is a prime ring because P is a prime ideal. Therefore A/P is simple by Corollary 1.4.8, whence P is maximal. □

In the following lemma we show that, in checking the structure of a Poisson module over a polynomial Poisson algebra, we only have to check the axioms on the generators.

Lemma 4.1.2. *Let $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$ with a Poisson bracket $\{-,-\}$. Let $V = \mathrm{Sp}(x_1, x_2, \ldots, x_n)$ and let M be an A-module. Suppose that there is a bilinear form $\{-,-\}_M : V \times M \to M$. Extend this to a bilinear form $\{-,-\}_M : A \times M \to M$ using Definition 1.9.8(ii) and $\{1, m\}_M = 0$. If Definition 1.9.8(i) and (iii) hold, for all $m \in M$, whenever $a = x_i$, and $a' = x_j$ for $1 \leq i < j \leq n$ then Definition 1.9.8(i) and (iii) hold for all $a, a' \in A$.*

Proof. The extension of $\{-,-\}_M$ from $V \times M$ to $A \times M$, using Definition 1.9.8(ii), is well-defined because A is free as a commutative algebra and is such that, for example,

$$\{x_1 x_2 \cdots x_n, m\}_M = \sum_{l=1}^{n} x_1 x_2 \cdots \widehat{x_l} \cdots x_n \{x_l, m\}_M,$$

where $\widehat{x_l}$ denotes omission of x_l. If Definition 1.9.8(i) and (iii) hold whenever $a = x_i$ and $a' = x_j$, $1 \leq i < j \leq n$, they hold whenever $a = x_i$ and $a' = x_j$ for any i and j because $\{x_i, x_i\} = 0$ and $\{x_j, x_i\} = -\{x_i, x_j\}$.

Let $a \in A$. Let $L(a') = \{a \in A : (i)$ holds for all $m \in M\}$ and $L = \{a' \in A : (i)$ holds for all $m \in M,\ a \in A\}$. We shall show that $L(a)$ is a subalgebra of A. Let $r, s \in L(a')$ and $m \in M$. Observe that

$$\begin{aligned} \{rs, a'm\}_M &= r\{s, a'm\}_M + s\{r, a'm\}_M \\ &= r(\{s, a'\}m + a'\{s, m\}_M) + s(\{r, a'\}m + a'\{r, m\}_M) \\ &= r\{s, a'\}m + ra'\{s, m\}_M + s\{r, a'\}m + sa'\{r, m\}_M, \end{aligned}$$

and

$$\begin{aligned}\{rs,a'\}m + a'\{rs,m\}_M &= (r\{s,a'\} + s\{r,a'\})m + a'(r\{s,m\}_M + s\{r,m\}_M) \\ &= r\{s,a'\}m + ra'\{s,m\}_M + s\{r,a'\}m + sa'\{r,m\}_M.\end{aligned}$$

Next, let $r, s \in L,\ m \in M$, and $a \in A$, we see that

$$\begin{aligned}\{a, rsm\}_M &= \{a,r\}sm + r\{a,sm\}_M \\ &= \{a,r\}sm + r(\{a,s\}m + s\{a,m\}_M) \\ &= \{a,r\}sm + r\{a,s\}m + rs\{a,m\}_M\end{aligned}$$

and

$$\begin{aligned}\{a,rs\}m + rs\{a,m\}_M &= (r\{a,s\} + \{a,r\}s)m + rs\{a,m\}_M \\ &= \{a,s\}rm + \{a,r\}sm + rs\{a,m\}_M \\ &= \{a, rsm\}_M.\end{aligned}$$

Therefore $rs \in L(a')$. By bilinearity, $L(a)$ is a subspace of A so $L(a)$ is also a subalgebra of A because the Poisson bracket $\{-,-\}$ is a bilinear form.

Let $R(a) = \{a' \in A : (iii) \text{ holds for all } m \in M\}$ and
$R = \{a \in A : (iii) \text{ holds for all } m \in M,\ a' \in A\}$. To show $R(a)$ is a subalgebra, let $r,\ s \in R(a)$ and $m \in M$, then

$$\begin{aligned}\{\{rs,a'\},m\}_M &= \{r\{s,a'\} + s\{r,a'\},m\}_M \\ &= \{r\{s,a'\},m\}_M + \{s\{r,a'\},m\}_M \\ &= r\{\{s,a'\},m\}_M + \{s,a'\}\{r,m\}_M \\ &\quad +s\{\{r,a'\},m\}_M + \{r,a'\}\{s,m\}_M \\ &= r(\{s,\{a',m\}_M\}_M - \{a',\{s,m\}_M\}_M) + \{s,a'\}\{r,m\}_M \\ &\quad +\{r,a'\}\{s,m\}_M + s(\{r,\{a',m\}_M\}_M - \{a',\{r,m\}_M\}_M),\end{aligned}$$

and $\{rs,\{a',m\}_M\}_M - \{a',\{rs,m\}_M\}_M$

$$\begin{aligned}&= r\{s,\{a',m\}_M\}_M + s\{r,\{a',m\}_M\}_M - \{a', r\{s,m\}_M + s\{r,m\}_M\}_M \\ &= r\{s,\{a',m\}_M\}_M + s\{r,\{a',m\}_M\}_M - \{a',r\}\{s,m\}_M \\ &\quad -r\{a',\{s,m\}_M\}_M - \{a',s\}\{r,m\}_M - s\{a',\{r,m\}_M\}_M.\end{aligned}$$

Therefore $rs \in R(a)$. By bilinearity, $R(a)$ is a subspace of A, so $R(a)$ is a subalgebra of A. Similarly, R is a subalgebra of A, the proof is similar to the proof for $R(a)$.

As $L(a)$ and $R(a)$ are subalgebras containing x_i for all $x_i, 1 \leq i \leq n$, we have $R(a) = A = L(a)$. Hence R and L are also subalgebras containing x_i for all i, $1 \leq i \leq n$, so $R = A = L$. □

Here is an example of quantization as defined in Definition 1.9.12.

Example 4.1.3. Let T be the $\mathbb{C}$-algebra generated by x, y, z, t and t^{-1} subject to the relations

$$\begin{aligned} xy - tyx &= (t-1)z, && (4.1)\\ yz - tzy &= (t-1)x, && (4.2)\\ zx - txz &= (t-1)y \quad \text{and} && (4.3) \end{aligned}$$

$$xt = tx, \quad yt = ty, \quad zt = tz, \quad tt^{-1} = 1 = t^{-1}t. \tag{4.4}$$

Then t is a central element of T. Let $A := T/(t-1)T \simeq \mathbb{C}[x, y, z]$ which is a commutative polynomial algebra. The induced Poisson bracket on A is such that

$$\{x, y\} = \tfrac{1}{t-1}[x, y] = \tfrac{1}{t-1}(xy - yx) = \tfrac{1}{t-1}(tyx + (t-1)z - yx) = yx + z.$$

Here we are abusing notation by writing x, y and z for both an element of T and its image in A. Similarly, we obtain

$$\{y, z\} = zy + x, \quad \{z, x\} = xz + y.$$

In the next lemma, we find the Poisson maximal ideals of A for this Poisson bracket.

Lemma 4.1.4. *In the above Poisson algebra A, there are only five Poisson maximal* *ideals of* A. *They are:*

$$\begin{aligned} J_1 &= xA + yA + zA, \\ J_2 &= (x+1)A + (y+1)A + (z+1)A, \\ J_3 &= (x+1)A + (y-1)A + (z-1)A, \\ J_4 &= (x-1)A + (y+1)A + (z-1)A \quad \textit{and} \\ J_5 &= (x-1)A + (y-1)A + (z+1)A. \end{aligned}$$

Proof. Let J be a Poisson maximal ideal of A. Since A is a commutative polynomial ring over $\mathbb{C}$, by Theorem 1.4.6, $J = (x-a, y-b, z-c)$ for suitable $a,\ b,\ c \in \mathbb{C}$. As J is Poisson, $\{x, J\} \subseteq J$, $\{y, J\} \subseteq J$, and $\{z, J\} \subseteq J$. Observe that

$$yx + z = \{x, y-b\} \in J, \tag{4.5}$$

$$-(zx + y) = \{x, z-c\} \in J, \tag{4.6}$$

$$zy + x = \{y, z-c\} \in J. \tag{4.7}$$

This happens precisely when $ab + c = ac + b = bc + a = 0$. As $c = -ba$, we have $0 = ca + b = -ba^2 + b$. This implies that $a = \pm 1$ or $b = 0$. Similarly $c = 0$ or $b = \pm 1$ and $a = 0$ or $c = \pm 1$. If $b = 0$ then $a = c = 0$ and, similarly, if $a = 0$ or $c = 0$ then $a = b = c = 0$. As $c = -ab$, there are five solutions:

(i) $a = b = c = 0$;

(ii) $a = b = c = -1$;

(iii) $a = b = 1$ and $c = -1$;

(iv) $b = c = 1$ and $a = -1$;

(v) $a = c = 1$ and $b = -1$.

□

In the next section, we will classify finite-dimensional simple Poisson A-modules annihilated by J_1.

4.2 Poisson modules annihilated by J_1

Lemma 4.2.1. *Let M be a Poisson module annihilated by $J_1 = xA + yA + zA$ and let $m \in M$. Then we have :*

(i) $xm = ym = zm = 0$.

(ii) $\{yx, m\}_M = \{zy, m\}_M = \{xz, m\}_M = 0$.

(iii) *(a)* $\{z, m\}_M = \{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M$;

(b) $\{x, m\}_M = \{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M$;

(c) $\{y, m\}_M = \{z, \{x, m\}_M\}_M - \{x, \{z, m\}_M\}_M$.

Proof. (i) It is obvious that $xm = ym = zm = 0$.

(ii) By (i) and Definition 1.9.8(ii), we have

$$\begin{aligned}
\{xy, m\}_M &= y\{x, m\}_M + x\{y, m\}_M = 0, \\
\{yz, m\}_M &= z\{y, m\}_M + y\{z, m\}_M = 0, \\
\{zx, m\}_M &= x\{z, m\}_M + z\{x, m\}_M = 0.
\end{aligned}$$

(iii) (a) By (ii) and Definition 1.9.8(iii),

$$\begin{aligned}
\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M &= \{\{x, y\}, m\}_M \\
&= \{xy, m\}_M + \{z, m\}_M = \{z, m\}_M.
\end{aligned}$$

(b) and (c) are proved similarly. □

Remark 4.2.2. Let M be a Poisson module annihilated by J_1. Let $m \in M$ be an eigenvector for $\{x, -\}_M$ with eigenvalue $\lambda \in \mathbb{C}$. Thus $\{x, m\}_M = \lambda m$. It follows from Lemma 4.2.1 (iii) that

(i) $\{x, \{y, m\}_M\}_M = \{z, m\}_M + \lambda\{y, m\}_M$;

(ii) $\{x, \{z, m\}_M\}_M = \lambda\{z, m\}_M - \{y, m\}_M$;

(iii) $\{y,\{z,m\}_M\}_M - \{z,\{y,m\}_M\}_M = \{x,m\}_M = \lambda m$.

To simplify these, we shall replace y and z by $u := \frac{1}{2}(y-iz)$ and $v := \frac{1}{2}(z-iy)$.

Lemma 4.2.3. *Let $A = \mathbb{C}[x,y,z]$ be equipped with the Poisson bracket*

$$\{x,y\} = yx + z, \quad \{y,z\} = zy + x, \quad \{z,x\} = xz + y.$$

If $u = \frac{1}{2}(y-iz)$, and $v = \frac{1}{2}(z-iy)$, then u, v, x generate A and the Poisson bracket is given by

$$\{x,v\} = -ixu - iv, \tag{4.8}$$

$$\{x,u\} = ixv + iu, \tag{4.9}$$

$$\{u,v\} = \tfrac{1}{2}(x + i(u^2+v^2)). \tag{4.10}$$

Proof. Since $iu = \frac{1}{2}(z+iy)$ and $iv = \frac{1}{2}(y+iz)$, we have $y = u+iv$ and $z = v+iu$, therefore u, v, x generate A. Firstly,

$$\begin{aligned}
\{x,v\} &= \{x,\tfrac{1}{2}(z-iy)\} = \tfrac{1}{2}\{x,z\} - \tfrac{1}{2}i\{x,y\} \\
&= \tfrac{1}{2}(-xz-y) - \tfrac{1}{2}i(yx+z) \\
&= -\tfrac{1}{2}x(z+iy) - \tfrac{1}{2}(y+iz) = -ixu - iv.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\{x,u\} &= \{x,\tfrac{1}{2}(y-iz)\} = \tfrac{1}{2}\{x,y\} - \tfrac{1}{2}i\{x,z\} \\
&= \tfrac{1}{2}x(y+iz) + \tfrac{1}{2}(z+iy) = ixv + iu.
\end{aligned}$$

Finally, we check that

$$\begin{aligned}
\{u,v\} &= \{\tfrac{1}{2}(y-iz), \tfrac{1}{2}(y-iz)\} \\
&= \tfrac{1}{2}\{y-iz, \tfrac{1}{2}z\} - \tfrac{1}{2}\{y-iz, \tfrac{i}{2}y\} = \tfrac{1}{4}\{y,z\} - \tfrac{1}{4}\{z,y\} \\
&= \tfrac{1}{2}\{zy+x\} = \tfrac{1}{2}((v+iu)(u+iv)+x) \\
&= \tfrac{1}{2}(x+i(u^2+v^2)).
\end{aligned}$$

Therefore the lemma is proved. $\square$

Lemma 4.2.4. *Let M be a Poisson module annihilated by J_1 and let $m \in M$. Then we have;*

(i) $xm = um = vm = 0$;

(ii) $\{xu, m\}_M = \{uv, m\}_M = \{xv, m\}_M = \{u^2, m\}_M = \{v^2, m\}_M = 0$;

(iii) *(a)* $\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = i\{u, m\}_M$;

(b) $\{x, \{v, m\}_M\}_M - \{v, \{x, m\}_M\}_M = -i\{v, m\}_M$;

(c) $\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \frac{1}{2}\{x, m\}_M$.

Proof. (i) It is easy to check that $xm = um = vm = 0$.

(ii) By (i) and Definition 1.9.8(ii), we have

$$\{xu, m\}_M = u\{x, m\}_M + x\{u, m\}_M = 0.$$

Similarly, we also obtain

$$\{uv, m\}_M = \{xv, m\}_M = \{u^2, m\}_M = 0 \text{ and } \{v^2, m\}_M = 0.$$

(iii) (a) By using (ii) and Definition 1.9.8 (iii),

$$\begin{aligned}\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M &= \{\{x, u\}, m\}_M = \{ixv + iu, m\}_M \\ &= i\{xv, m\}_M + i\{u, m\}_M = i\{u, m\}_M.\end{aligned}$$

(b) and (c) are proved similarly. $\square$

Lemma 4.2.5. *Let $u = \frac{1}{2}(y - iz)$ and $v = \frac{1}{2}(z - iy)$ (so that $y = u + iv$ and $z = v + iu$). Let M be a Poisson module annihilated by $J := J_1$. Let $\lambda \in \mathbb{C}$ be such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. Then*

(i) $\{x, \{v, m\}_M\}_M = (\lambda - i)\{v, m\}_M$,

(ii) $\{x, \{u, m\}_M\}_M = (\lambda + i)\{u, m\}_M$,

(iii) $\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = \frac{1}{2}\lambda m$.

Proof. (i) By Lemma 4.2.4(iii)(b), we have

$$\begin{aligned}\{x, \{v, m\}_M\}_M &= \{v, \{x, m\}_M\}_M - i\{v, m\}_M \\ &= \lambda\{v, m\}_M - i\{v, m\}_M = (\lambda - i)\{v, m\}_M.\end{aligned}$$

(ii) We obtain this from Lemma 4.2.4(iii)(a) by the same method as in (i).

(iii) This is immediate from Lemma 4.2.4(iii)(c). □

Lemma 4.2.6. *Let $A = \mathbb{C}[x, u, v]$ with the Poisson bracket as in Lemma 4.2.3. Let $d \geq 1$. There is a d-dimensional Poisson A-module M, with basis $\{m_1, m_2, \ldots, m_d\}$, such that $xM = vM = uM = 0$ and*

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ *for* $1 \leq j \leq d$

(ii) $\{v, m_1\}_M = 0$ *and* $\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ *for* $1 < j \leq d$

(iii) $\{u, m_j\}_M = m_{j+1}$ *for* $1 \leq j < d$ *and* $\{u, m_d\}_M = 0$,

where $\lambda = \frac{1-d}{2}i$.

Proof. By Lemma 4.1.2, it is enough to show that Definition 1.9.8(i) and (iii) hold when $m = m_j$ and $(a, a') = (x, u), (x, v)$ or (u, v) for the brackets defined by (i), (ii) and (iii) above. We then extend the Poisson action on M from $V := \mathbb{C}x + \mathbb{C}u + \mathbb{C}v$ to $\mathbb{C}[x, u, v]$ using Definition 1.9.8(ii). Then $\{V^2, m\}_M = 0$ and the conclusion of Lemma 4.2.4(i) holds.

We shall show that Definition 1.9.8(i) holds for m and (a, a') as defined above.

In the case, $1 \leq j < d$. We first check that, by Lemma 4.2.4(i), $\{x, um_j\}_M = 0$ and

$$\{x, u\}m_j + \{x, um_j\}_M = -(ixu + iv)m_j = 0 = \{x, um_j\}_M.$$

We compute that $\{x, vm_j\}_M = 0 = \{x, v\}m_j + \{x, vm_j\}_M$ and

$$\{u, vm_j\}_M = 0 = \{u, v\}m_j + \{u, vm_j\}_M.$$

In the case, $j = d$. We see that, by Lemma 4.2.4(i),

$$\begin{aligned}
\{x, um_d\}_M &= 0 = \{x, u\}m_d + \{x, um_d\}_M \\
\{x, vm_d\}_M &= 0 = \{x, v\}m_d + \{x, vm_d\}_M \\
\{u, vm_d\}_M &= 0 = \{u, v\}m_d + \{u, vm_d\}_M
\end{aligned}$$

Next, we check that Definition 1.9.8(iii) holds for m and (a, a') as defined above.

Firstly, we show that $\{\{x, u\}, m_j\}_M = \{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M$ for $1 \leq j \leq d$. Let us consider the case when $1 \leq j < d$. By Lemma 4.2.3,

$$\{\{x, u\}, m_j\}_M = \{ixv, m_j\}_M + \{iu, m_j\}_M = i\{u, m_j\}_M = im_{j+1} \tag{4.11}$$

and

$$\begin{aligned}
\{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M &= \{x, m_{j+1}\}_M - \{u, (\lambda + (j-1)i)m_j\}_M \\
&= (\lambda + ji)m_{j+1} - (\lambda + (j-1)i)m_{j+1} \\
&= im_{j+1} = \{\{x, u\}, m_j\}_M.
\end{aligned}$$

Next, by (4.11), we have $\{\{x, u\}, m_d\}_M = i\{u, m_d\}_M = 0$ and also

$$\{x, \{u, m_d\}_M\}_M - \{u, \{x, m_d\}_M\}_M = (\lambda + (j-1)i)\{u, m_d\}_M = 0.$$

It follows that $\{\{x, u\}, m_j\}_M = \{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M$ for $1 \leq j \leq d$.

Secondly, we show that $\{\{x, v\}, m_j\}_M = \{x, \{v, m_j\}_M\}_M - \{v, \{x, m_j\}_M\}_M$ for $1 \leq j \leq d$. In the case $1 \leq j < d$, we compute that, by Lemma 4.2.3,

$$\{\{x, v\}, m_j\}_M = -\{iv + ixu, m_j\}_M = -i\{v, m_j\}_M = \tfrac{i}{2}(j-1)(\lambda + \tfrac{1}{2}(j-2)i)m_{j-1}$$

and

$$\begin{aligned}
&\{x, \{v, m_j\}_M\}_M - \{v, \{x, m_j\}_M\}_M \\
&\quad= -\tfrac{1}{2}(j-1)(\lambda + \tfrac{1}{2}(j-2)i)\{x, m_{j-1}\}_M - (\lambda + (j-1)i)\{v, m_j\}_M \\
&\quad= -\tfrac{1}{2}(j-1)(\lambda + \tfrac{1}{2}(j-2)i)(\lambda + (j-2)i)m_{j-1} \\
&\qquad+\tfrac{1}{2}(j-1)(\lambda + \tfrac{1}{2}(j-2)i)(\lambda + (j-1)i)m_{j-1} \\
&\quad= \tfrac{i}{2}(j-1)(\lambda + \tfrac{1}{2}(j-2)i)m_{j-1} = \{\{x, v\}, m_j\}_M.
\end{aligned}$$

In the case $j = d$, we check that

$$\begin{aligned} \{\{x,v\},m_d\}_M &= -\{iv+ixu,m_d\}_M = -i\{v,m_d\}_M \quad \text{by Lemma 4.2.3} \\ &= \tfrac{i}{2}(d-1)(\lambda+\tfrac{1}{2}(d-2)i)m_{d-1} \end{aligned}$$

and $\{x,\{v,m_d\}_M\}_M - \{v,\{x,m_d\}_M\}_M$

$$\begin{aligned} &= -\tfrac{1}{2}(d-1)(\lambda+\tfrac{1}{2}(d-2)i)\{x,m_{d-1}\}_M - (\lambda+(d-1)i)\{v,m_d\}_M \\ &= -\tfrac{1}{2}(d-1)(\lambda+\tfrac{1}{2}(d-2)i)(\lambda+(d-2)i)m_{d-1} \\ &\quad +\tfrac{1}{2}(\lambda+(d-1)i)(d-1)(\lambda+\tfrac{1}{2}(d-2)i)m_{d-1} \\ &= \tfrac{1}{2}i(d-1)(\lambda+\tfrac{1}{2}(d-2)i)m_{d-1} \\ &= \{\{x,v\},m_d\}_M. \end{aligned}$$

We next show that for $1 \le j < d$,

$$\{\{u,v\},m_j\}_M = \{u,\{v,m_j\}_M\}_M - \{v,\{u,m_j\}_M\}_M.$$

Here, by Lemma 4.2.3,

$$\{\{u,v\},m_j\}_M = \tfrac{1}{2}\{x+i(u^2+v^2),m_j\}_M = \tfrac{1}{2}\{x,m_j\}_M = \tfrac{1}{2}(\lambda+(j-1)i)m_j,$$

and

$$\begin{aligned} &\{u,\{v,m_j\}_M\}_M - \{v,\{u,m_j\}_M\}_M \\ &\quad= -\tfrac{1}{2}(j-1)(\lambda+\tfrac{1}{2}(j-2)i)\{u,m_{j-1}\}_M - \{v,m_{j+1}\}_M \\ &\quad= -\tfrac{1}{2}(j-1)(\lambda+\tfrac{1}{2}(j-2)i)m_j + \tfrac{1}{2}j(\lambda+\tfrac{1}{2}(j-1)i)m_j \\ &\quad= \tfrac{1}{2}(\lambda+(j-1)i)m_j = \{\{u,v\},m_j\}_M. \end{aligned}$$

In the case $j = d$ and $\lambda = \frac{d-1}{2}i$, by Lemma 4.2.3,

$$\begin{aligned} \{\{u,v\},m_d\}_M &= \tfrac{1}{2}\{x+i(u^2+v^2),m_d\}_M \\ &= \tfrac{1}{2}\{x,m_d\}_M = \tfrac{1}{2}(\lambda+(d-1)i)m_d \\ &= \tfrac{1}{2}(\tfrac{d-1}{2}i+(d-1)i)m_d = \tfrac{i}{4}(d-1)m_d, \end{aligned}$$

and

$$\begin{aligned}\{u,\{v,m_d\}_M\}_M-\{v,\{u,m_d\}_M\}_M &= -\tfrac{1}{2}(d-1)(\lambda+\tfrac{1}{2}(d-2)i)\{u,m_{d-1}\}_M\\ &= -\tfrac{1}{2}(d-1)(\tfrac{d-1}{2}i+\tfrac{1}{2}(d-2)i)m_d\\ &= \tfrac{i}{4}(d-1)m_d.\end{aligned}$$

□

Lemma 4.2.7. *Let $d \geq 1$. The d-dimensional Poisson module constructed in Lemma 4.2.6 is simple as a Poisson module.*

Proof. Let $\lambda_j = \lambda + (j-1)i$, $1 \leq j \leq d$. Note that $\lambda_j \neq \lambda_k$ when $j \neq k$. Let N be a non-zero Poisson submodule of M. Let $0 \neq n = \sum_{j=1}^{d} \alpha_j m_j \in N$ be such that minimally many of the coefficients $\alpha_j \in \mathbb{F}$ are non-zero and choose k so that $\alpha_k \neq 0$.

$$\{x,n\}_M - \lambda_k n = \sum_{j=1}^{d} \alpha_j(\lambda_j - \lambda_k)m_j.$$

This has one fewer non-zero coefficient than n so, by minimality, it is 0 and hence $\alpha_j = 0$ when $j \neq k$, that is $n = \alpha_k m_k$. Therefore $m_k \in N$. By the Poisson action of u and v, $m_j \in N$ for all j. So $N = M$ and M is a simple Poisson module. □

We now aim to show that the simple d-dimensional Poisson module constructed above is unique.

Lemma 4.2.8. *Let M be a finite-dimensional simple Poisson module annihilated by $J_1 = uA + vA + xA$ and let $n \leq \dim M$. There exist $\lambda \in \mathbb{C}$ and n linearly independent elements $m_1, m_2, \ldots, m_n \in M$ such that*

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ *for* $1 \leq j \leq n$,

(ii) $\{v, m_1\}_M = 0$ *and* $\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ *for* $1 < j \leq n$,

(iii) $\{u, m_j\}_M = m_{j+1}$ *for* $1 \leq j < n$.

Proof. Let $\Lambda = \{\lambda \in \mathbb{C} : \{x, m\}_M = \lambda m$ for some $0 \neq m \in M\}$. Since $\dim_{\mathbb{F}} M < \infty$ the linear transformation of M, with $m \mapsto \{x, m\}_M$, has an eigenvalue, therefore $\Lambda \neq \emptyset$. As x, u, v generate A, it follows from Lemma 4.2.4(iii) and Lemma 4.2.5(ii) that if $\lambda \in \Lambda$ then $\{m \in M : \{x, m\}_M = (\lambda + ni)m$ for some $n \in \mathbb{Z}\}$ spans a non-zero Poisson module of M. As M is finite-dimensional, $\lambda \in \Lambda$ can be chosen so that $\lambda - i \notin \Lambda$. Let m_1 be an eigenvector for $\{x, -\}_M$ with eigenvalue λ. By Lemma 4.2.5(ii), $\{x, \{v, m_1\}_M\}_M = (\lambda - i)\{v, m_1\}_M$ so $\{v, m_1\}_M = 0$. Thus the result is true when $n = 1$. We proceed by induction on n. Suppose that (i), (ii) and (iii) hold for n and that $n + 1 \leq \dim M$. Then $\{u, m_n\}_M \neq 0$, otherwise $\mathrm{Sp}(m_1, m_2, \dots, m_n)$ is an n-dimensional Poisson submodule of M, contrary to the Poisson simplicity of M. Let $m_{n+1} = \{u, m_n\}_M$. By Lemma 4.2.5(ii) $\{x, m_n\}_M = (\lambda + (n-1)i)m_n$ and $\{x, m_{n+1}\}_M = \{x, \{u, m_n\}_M\}_M = (\lambda + ni)m_{n+1}$, in accordance with (iii) for $j = n + 1$. By Lemma 4.2.5(iii),

$$\begin{aligned}
\{u, \{v, m_n\}_M\}_M - \{v, \{u, m_n\}_M\}_M &= \tfrac{1}{2}\{x, m_n\}_M, \text{ so} \\
-\tfrac{1}{2}(n-1)(\lambda + \tfrac{1}{2}(n-2)i)\{u, m_{n-1}\}_M - \{v, m_{n+1}\}_M &= \tfrac{1}{2}(\lambda + (n-1)i)m_n
\end{aligned}$$

and hence,

$$\{v, m_{n+1}\}_M = -\tfrac{1}{2}(n-1)(\lambda + \tfrac{1}{2}(n-2)i)m_n - \tfrac{1}{2}(\lambda + (n-1)i)m_n.$$

It follows that

$$\{v, m_{n+1}\}_M = -\tfrac{1}{2}n(\lambda + \tfrac{1}{2}(n-1)i)m_n.$$

Note that, being eigenvectors for $\{x, -\}_M$ with distinct eigenvalues, $m_1, \dots, m_{n+1}$ are linearly independent. The result holds by induction on n. □

Theorem 4.2.9. *Let M be a finite-dimensional simple Poisson module annihilated by $J_1 = xA + uA + vA$ and let $d = \dim_{\mathbb{C}} M$. There exist d linearly independent element $m_1, \dots, m_d \in M$ such that*

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ *for* $1 \leq j \leq d$.

(ii) $\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ *for* $1 \leq j \leq d$.

(iii) $\{u, m_j\}_M = m_{j+1}$ *for* $1 \leq j < d$ *and* $\{u, m_d\}_M = 0$,

where $\lambda = -\frac{d-1}{2}i$.

Proof. By Lemma 4.2.8, there exist $\lambda \in \mathbb{C}$ and linearly independent elements $m_1, \ldots, m_d$ of M such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \leq j \leq d$,

(ii) $\{v, m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ for $1 \leq j \leq d$ and

(iii) $\{u, m_j\}_M = m_{j+1}$ for $1 \leq j < d$.

As $\dim M = d$, $M = \mathrm{Sp}(m_1, m_2, ..., m_d)$ and is the sum of the eigenspaces for $\{x, -\}_M$ and the eigenvalues $\lambda, \lambda + i, \ldots, \lambda + (d-1)i$. By Lemma 4.2.5,

$$\{x, \{u, m_d\}_M\}_M = (\lambda + di)\{u, m_d\}_M$$

but $\lambda + di$ is not an eigenvalue of $\{u, m_d\}_M$ for $\{x, -\}_M$ so $\{u, m_d\}_M = 0$. By Lemma 4.2.5 (iii),

$$\begin{aligned} \{u, \{v, m_d\}_M\}_M - \{v, \{u, m_d\}_M\}_M &= \tfrac{1}{2}\{x, m_d\}_M \\ -\tfrac{1}{2}(\lambda + \tfrac{1}{2}(d-2)i)\{u, m_{d-1}\}_M &= \tfrac{1}{2}(\lambda + (d-1)i)m_d \\ -\tfrac{1}{2}(\lambda + \tfrac{1}{2}(d-2)i)m_d &= \tfrac{1}{2}(\lambda + (d-1)i)m_d \\ d(\lambda + \tfrac{1}{2}(d-1)i) = 0 & \end{aligned}$$

from which it follows that $\lambda = -\frac{1}{2}(d-1)i$. □

We conclude from Lemma 4.2.6 and Theorem 4.2.9 that for $d \geq 1$ there is a unique $d-$dimensional simple Poisson module M annihilated by J_1.

Here are two examples.

Example 4.2.10. Let $A = \mathbb{C}[x, u, v]$ with the Poisson bracket as in Lemma 4.2.3. The matrices representing the action $\{x, -\}_M, \{u, -\}_M$ and $\{v, -\}_M$ on the 3-dimensional and 4-dimensional simple Poisson modules, with respect to the basis in Theorem 4.2.9, are shown below:

3-dimensional simple Poisson module:

$$\{x,-\}_M = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix},\ \{u,-\}_M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \{v,-\}_M = \begin{pmatrix} 0 & \frac{1}{2}i & 0 \\ 0 & 0 & \frac{1}{2}i \\ 0 & 0 & 0 \end{pmatrix}$$

4-dimensional simple Poisson module:

$$\{x,-\}_M = \begin{pmatrix} -\frac{3}{2}i & 0 & 0 & 0 \\ 0 & -\frac{1}{2}i & 0 & 0 \\ 0 & 0 & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & \frac{3}{2}i \end{pmatrix},\ \{u,-\}_M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $\{v,-\}_M = \begin{pmatrix} 0 & \frac{3}{4}i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \frac{3}{4}i \\ 0 & 0 & 0 & 0 \end{pmatrix}$

We have expressed the unique d-dimensional simple Poisson module annihilated by J_1 for each $d \geq 1$ in terms of the action of $x,\ u,\ v$. The following theorem gives us the same result with the action of $x,\ y,\ z$.

Theorem 4.2.11. *Let $d \geq 1$. There is a unique d-dimensional simple Poisson module over A, annihilated by J_1. It has a basis $m_1, m_2, \ldots, m_d$ such that*

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ *for* $1 \leq j \leq d$,

(ii) $\{y, m_1\}_M = m_2$,
$\{y, m_j\}_M = m_{j+1} - \frac{1}{2}i(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ *for* $1 < j < d$ *and*
$\{y, m_d\}_M = -\frac{1}{2}i(d-1)(\lambda + \frac{1}{2}(d-2)i)m_{d-1}$

(iii) $\{z, m_1\}_M = im_2$,
$\{z, m_j\}_M = im_{j+1} - \frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ *for* $1 < j < d$ *and*
$\{z, m_d\}_M = -\frac{1}{2}(d-1)(\lambda + \frac{1}{2}(d-2)i)m_{d-1}$,

where $\lambda = \frac{1-d}{2}i$.

Proof. This is immediate from Lemma 4.2.3, Lemma 4.2.6 and Theorem 4.2.9. □

4.3 Poisson modules annihilated by J_2

We now consider Poisson modules annihilated by $J_2 = (x+1)A+(y+1)A+(z+1)A$.

Lemma 4.3.1. *Let M be a Poisson module annihilated by*

$$J_2 = (x+1)A + (y+1)A + (z+1)A$$

and let $m \in M$. Then we have :

(i) $xm = ym = zm = -m$.

(ii) *(a)* $\{xy, m\}_M = -\{y, m\}_M - \{x, m\}_M$;

(b) $\{yz, m\}_M = -\{y, m\}_M - \{z, m\}_M$;

(c) $\{xz, m\}_M = -\{z, m\}_M - \{x, m\}_M$.

(iii) *(a)* $\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M = \{z, m\}_M - \{x, m\}_M - \{y, m\}_M$;

(b) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \{x, m\}_M - \{y, m\}_M - \{z, m\}_M$;

(c) $\{z, \{x, m\}_M\}_M - \{x, \{z, m\}_M\}_M = \{y, m\}_M - \{x, m\}_M - \{z, m\}_M$.

Proof. (i) It is obvious that $xm = ym = zm = -m$.

(ii) By (i) and Definition 1.9.8(ii), we have

(a) $\{xy, m\}_M = x\{y, m\}_M + y\{x, m\}_M = -\{y, m\}_M - \{x, m\}_M$.

(b) $\{yz, m\}_M = y\{z, m\}_M + z\{y, m\}_M = -\{z, m\}_M - \{y, m\}_M$.

(c) $\{zx, m\}_M = x\{z, m\}_M + z\{x, m\}_M = -\{z, m\}_M - \{x, m\}_M$.

(iii) (a) By (ii) and Definition 1.9.8(iii),

$$\begin{aligned} \{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M &= \{\{x, y\}, m\}_M = \{yx + z, m\}_M \\ &= \{z, m\}_M - \{x, m\}_M - \{y, m\}_M \end{aligned}$$

(b) and (c) are proved similarly.

□

Remark 4.3.2. Let M be a Poisson module annihilated by J_2 and let $m \in M$ be an eigenvector for $\{x, -\}_M$ with eigenvalue $\lambda \in \mathbb{C}$. Then $\{x, m\}_M = \lambda m$. It follows from Lemma 4.3.1(iii) that

(i) $\{x, \{y, m\}_M\}_M = (\lambda - 1)\{y, m\}_M + \{z, m\}_M - \lambda m$;

(ii) $\{x, \{z, m\}_M\}_M = \{y, m\}_M + \lambda m + (\lambda + 1)\{z, m\}_M$;

(iii) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = \lambda m - \{y, m\}_M - \{z, m\}_M$.

To simplify these, we shall replace z by $u := z - x - y$.

Lemma 4.3.3. *Let $A = \mathbb{C}[x, y, u]$ be equipped with the Poisson bracket*

$$\{x, y\} = yx + z, \quad \{y, z\} = zy + x, \quad \{z, x\} = xz + y.$$

If $u = z - x - y$ then x, y, u generate A and the Poisson bracket is given by

(i) $\{x, y\} = (x + 1)y + u + x$;

(ii) $\{x, u\} = -(x + 1)(2y + x + u)$;

(iii) $\{y, u\} = (y + 1)(u + 2x + y)$.

Proof. Since $u = z - x - y$, we have $z = x + y + u$ and therefore x, y, u generate A.

(i) $\{x, y\} = yx + z = yx + x + y + u = (x + 1)y + u + x$.

(ii)

$$\begin{aligned}
\{x, u\} &= \{x, z\} - \{x, y\} \\
&= -xz - y - yx - z \\
&= -(x + 1)(y + z) \\
&= -(x + 1)(2y + x + u).
\end{aligned}$$

(iii)

$$\begin{aligned}\{y,u\} &= \{y,z\}-\{y,x\}\\ &= zy+x+yx+z\\ &= (y+1)(z+x)\\ &= (y+1)(2x+y+u).\end{aligned}$$

□

Lemma 4.3.4. *Let M be a Poisson module annihilated by*

$$J_2 = (x+1)A + (y+1)A + (u-1)A$$

and let $m \in M$. Then we have :

(i) $xm = ym = -m$, *and* $um = m$;

(ii) *(a)* $\{xy, m\}_M = -\{y,m\}_M - \{x,m\}_M$;

(b) $\{xu, m\}_M = -\{u,m\}_M + \{x,m\}_M$;

(c) $\{yu, m\}_M = -\{u,m\}_M + \{y,m\}_M$.

(iii) *(a)* $\{x, \{y,m\}_M\}_M - \{y, \{x,m\}_M\}_M = \{u,m\}_M$;

(b) $\{x, \{u,m\}_M\}_M - \{u, \{x,m\}_M\}_M = 2\{x,m\}_M$;

(c) $\{y, \{u,m\}_M\}_M - \{u, \{y,m\}_M\}_M = -2\{y,m\}_M$.

Proof. (i) It is obvious that $xm = ym = -m$ and $um = m$.

(ii) By (i) and Definition 1.9.8(ii), we have

(a) $\{xy, m\}_M = x\{y,m\}_M + y\{x,m\}_M = -\{y,m\}_M - \{x,m\}_M$

(b) $\{xu, m\}_M = x\{u,m\}_M + u\{x,m\}_M = -\{u,m\}_M + \{x,m\}_M$;

(c) $\{yu, m\}_M = y\{u,m\}_M + u\{y,m\}_M = -\{u,m\}_M + \{y,m\}_M$.

(iii) (a) By (ii) and Definition 1.9.8(iii), we have

$$\begin{aligned}\{x,\{y,m\}_M\}_M - \{y,\{x,m\}_M\}_M & \\ &= \{\{x,y\},m\}_M = \{xy,m\}_M + \{z,m\}_M \\ &= -\{y,m\}_M - \{x,m\}_M + \{z,m\}_M \\ &= \{z-x-y,m\}_M = \{u,m\}_M.\end{aligned}$$

(b) and (c) are proved similarly.

□

Lemma 4.3.5. *Let $u = z - y - x$ (so that $z = u + x + y$). Let M be a finite-dimensional simple Poisson module annihilated by J_2. Let $\lambda \in \mathbb{C}$ be such that $\{u,m\}_M = \lambda m$ for some $0 \neq m \in M$. Then*

(i) $\{x,\{y,m\}_M\}_M - \{y,\{x,m\}_M\}_M = \lambda m$.

(ii) $\{u,\{x,m\}_M\}_M = (\lambda - 2)\{x,m\}_M$.

(iii) $\{u,\{y,m\}_M\}_M = (\lambda + 2)\{y,m\}_M$.

Proof. These are easily checked from Lemma 4.3.4(iii). □

Lemma 4.3.6. *Let $A = \mathbb{C}[x,y,u]$ with the Poisson bracket as in Lemma 4.3.3. Let $d \geq 1$. There is a d-dimensional Poisson A-module M, with a basis $\{m_1, m_2, \ldots, m_d\}$, such that $(x+1)M = (y+1)M = (u-1)M = 0$ and*

(i) $\{u,m_j\}_M = (\lambda + 2(j-1))m_j$ *for* $1 \leq j \leq d$,

(ii) $\{x,m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1}$ *for* $1 \leq j \leq d$,

(iii) $\{y,m_j\}_M = m_{j+1}$ *for* $1 \leq j < d$ *and* $\{y,m_d\}_M = 0$,

where $\lambda = 1 - d$.

Proof. By Lemma 4.1.2, it is enough to show that Definition 1.9.8(i) and (iii) hold for $m = m_j$ and $(a, a') = (x, u), (y, u)$ or (x, y) for the brackets defined (i), (ii) and (iii) above. We then extend the Poisson action on M from $V := \mathbb{C}x + \mathbb{C}y + \mathbb{C}u$ to $\mathbb{C}[x, y, u]$ using Definition 1.9.8(ii). Then the conclusions of Lemma 4.3.4(i) and (ii) hold and we also have $xym = m$, $xum = yum = -m$.

To show Definition 1.9.8(iii) holds for $m = m_j$ and (a, a') defined as above, we first show that

$$\{\{x, y\}, m_j\}_M = \{x, \{y, m_j\}_M\}_M - \{y, \{x, m_j\}_M\}_M \text{ for } 1 \leq j < d.$$

Here

$$\begin{aligned}
\{\{x, y\}, m_j\}_M &= \{yx + u + x + y, m_j\}_M \quad \text{by Lemma 4.3.3} \\
&= -\{x, m_j\}_M - \{y, m_j\}_M + \{u, m_j\}_M + \{x, m_j\}_M + \{y, m_j\}_M \\
&= \{u, m_j\}_M = (\lambda + 2(j-1))m_j,
\end{aligned}$$

and

$$\begin{aligned}
&\{x, \{y, m_j\}_M\}_M - \{y, \{x, m_j\}_M\}_M \\
&\quad = \{x, m_{j+1}\}_M - (j-1)(\lambda + j - 2)\{y, m_{j-1}\}_M \\
&\quad = j(\lambda + j - 1)m_j - (j-1)(\lambda + j - 2)m_j \\
&\quad = (\lambda + 2(j-1))m_j = \{\{x, y\}, m_j\}_M.
\end{aligned}$$

In the case $j = d$, observe that

$$\begin{aligned}
\{\{x, y\}, m_d\}_M &= \{u, m_d\}_M = (\lambda + 2(d-1))m_d \\
&= (1 - d + 2(d-1))m_d \\
&= (d-1)m_d \text{ where } \lambda = 1 - d,
\end{aligned}$$

and

$$\begin{aligned}
\{x, \{y, m_d\}_M\}_M - \{y, \{x, m_d\}_M\}_M &= -(d-1)(\lambda + d - 2)\{y, m_{d-1}\}_M \\
&= -(d-1)(\lambda + d - 2)m_d \text{ where } \lambda = 1 - d \\
&= (d-1)m_d = \{\{x, y\}, m_d\}_M.
\end{aligned}$$

Secondly, we show that

$$\{\{x, u\}, m_j\}_M = \{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M \text{ for } 1 \leq j \leq d.$$

In the case $1 \leq j < d$, we compute that, by Lemma 4.3.3,

$$\begin{aligned}
\{\{x, u\}, m_j\}_M &= -\{2y + x + u, m_j\}_M - \{2xy + x^2 + xu, m_j\}_M \\
&= -\{x, m_j\}_M - \{u, m_j\}_M - 2(-\{x, m_j\}_M - \{y, m_j\}_M) \\
&\quad -2\{y, m_j\}_M + 2\{x, m_j\}_M + \{u, m_j\}_M - \{x, m_j\}_M \\
&= 2\{x, m_j\}_M = 2(j-1)(\lambda + j - 2)m_{j-1},
\end{aligned}$$

and $\{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M$

$$\begin{aligned}
&= (\lambda + 2(j-1))\{x, m_j\}_M - (j-1)(\lambda + j - 2)\{u, m_{j-1}\}_M \\
&= (\lambda + 2(j-1))(\lambda + j - 2)m_{j-1} \\
&\quad -(j-1)(\lambda + j - 2)(\lambda + 2(j-2))m_{j-1} \\
&= 2(j-1)(\lambda + j - 2)m_{j-1} = \{\{x, u\}, m_j\}_M.
\end{aligned}$$

In the case $j = d$, we find that

$$\{\{x, u\}, m_d\}_M = 2\{x, m_d\}_M = 2(d-1)(\lambda + d - 2)m_{d-1},$$

and $\{x, \{u, m_d\}_M\}_M - \{u, \{x, m_d\}_M\}_M$

$$\begin{aligned}
&= (\lambda + 2(d-1))\{x, m_d\}_M - (d-1)(\lambda + d - 2)\{u, m_{d-1}\}_M \\
&= (d-1)(\lambda + d - 2)(\lambda + 2(d-1))m_{d-1} \\
&\quad -(d-1)(\lambda + d - 2)(\lambda + 2(d-2))m_{d-1} \\
&= 2(d-1)(\lambda + d - 2)m_{d-1} = \{\{x, u\}, m_d\}_M.
\end{aligned}$$

Finally, we shall show that

$$\{\{y, u\}, m_j\}_M = \{y, \{u, m_j\}_M\}_M - \{u, \{y, m_j\}_M\}_M \text{ for } 1 \leq j \leq d.$$

In the case $1 \leq j < d$, we see that, by Lemma 4.3.3,

$$\begin{aligned}
\{\{y,u\},m_j\}_M &= \{uy+2xy+y^2+u+2x+y,m_j\}_M \\
&= -\{u,m_j\}_M+\{y,m_j\}_M+2(-\{y,m_j\}_M-\{x,m_j\}_M) \\
&\quad -2\{y,m_j\}_M+\{u,m_j\}_M+2\{x,m_j\}_M+\{y,m_j\}_M \\
&= -2\{y,m_j\}_M=-2m_{j+1},
\end{aligned}$$

and $\{y,\{u,m_j\}_M\}_M-\{u,\{y,m_j\}_M\}_M$

$$\begin{aligned}
&= (\lambda+2(j-1))\{y,m_j\}_M-\{u,m_j\}_M \\
&= (\lambda+2(j-1))m_{j+1}-(\lambda+2j)m_{j+1} \\
&= -2m_{j+1}=\{\{y,u\},m_j\}_M.
\end{aligned}$$

In the case $j=d$, we observe that $\{\{y,u\},m_d\}_M=-2\{y,m_d\}_M=0$ and

$$\begin{aligned}
\{y,\{u,m_d\}_M\}_M-\{u,\{y,m_d\}_M\}_M &= (\lambda+2(d-1))\{y,m_d\}_M \\
&= 0=\{\{y,u\},m_d\}_M.
\end{aligned}$$

It remains to show Definition 1.9.8(i) holds for m and (a,a') as defined above.

Case: $1 \leq j < d$. By Lemma 4.3.4, we first check that

$$\{x,ym_j\}_M=-\{x,m_j\}_M=-(j-1)(\lambda+j-2)m_{j-1}$$

and

$$\begin{aligned}
\{x,y\}m_j+y\{x,m_j\}_M &= (xy+y+u+x)m_j-\{x,m_j\}_M \\
&= -(j-1)(\lambda+j-2)m_{j-1} \\
&= \{x,ym_j\}_M.
\end{aligned}$$

Secondly, we check that $\{x,um_j\}_M=\{x,m_j\}_M=(j-1)(\lambda+j-2)m_{j-1}$ and

$$\begin{aligned}
\{x,u\}m_j+u\{x,m_j\}_M &= -(x+1)(2y+x+u)m_j+\{x,m_j\}_M \\
&= (j-1)(\lambda+j-2)m_{j-1} \\
&= \{x,um_j\}_M.
\end{aligned}$$

Finally, $\{y, um_j\}_M = \{y, m_j\}_M = m_{j+1}$ and

$$\begin{aligned}\{y,u\}m_j + u\{y,m_j\}_M &= (y+1)(u+2x+y)m_j + \{y,m_j\}_M \\ &= m_{j+1} = \{y, um_j\}_M.\end{aligned}$$

Next, we consider the case $j = d$. Firstly, we see that

$$\{x, ym_d\}_M = -\{x, m_d\}_M = -(d-1)(\lambda+d-2)m_{d-1}$$

and

$$\begin{aligned}\{x,y\}m_d + y\{x,m_d\}_M &= (xy+y+u+x)m_d - \{x,m_d\}_M \\ &= -(d-1)(\lambda+d-2)m_{d-1} \\ &= \{x, ym_d\}_M.\end{aligned}$$

Secondly, $\{x, um_d\}_M = \{x, m_d\}_M = (d-1)(\lambda+d-2)m_{d-1}$ and

$$\begin{aligned}\{x,u\}m_d + u\{x,m_d\}_M &= -(x+1)(2y+x+u)m_d + \{x,m_d\}_M \\ &= (d-1)(\lambda+d-2)m_{d-1} = \{x, um_d\}_M.\end{aligned}$$

Also, we see that $\{y, um_d\}_M = \{y, m_d\}_M = 0$ and $\{y,u\}m_d + u\{y,m_d\}_M = (y+1)(u+2x+y)m_d + \{y,m_d\}_M = 0$. Therefore the lemma is proved. □

The d-dimensional Poisson module constructed in Lemma 4.3.6 is simple. The proof is similar to the proof for a d-dimensional simple Poisson module annihilated by J_1 in Section 4.2.

Our next aim is to show that the d-dimensional simple Poisson module constructed as above is unique.

Lemma 4.3.7. *Let M be a finite-dimensional simple Poisson module annihilated by J_2 and let $n \leq \dim_{\mathbb{C}} M$. Then there exist $\lambda \in \mathbb{C}$ and n linearly independent elements $m_1, \ldots, m_n \in M$ such that*

(i) $\{u, m_j\}_M = (\lambda + 2(j-1))m_j$ *for* $1 \leq j \leq n$,

(ii) $\{x, m_1\}_M = 0$ *and* $\{x, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1}$ *for* $1 < j \leq n$,

(iii) $\{y, m_j\}_M = m_{j+1}$ *for* $1 \leq j < n$,

where $\lambda = 1 - d$.

Proof. The proof is similar to the proof of Lemma 4.2.8, using the generators x, y and u, with u in the role of x in Lemma 4.2.8, defining

$$\Lambda = \{\lambda \in \mathbb{C} : \{u, m\}_M = \lambda m \text{ for some } 0 \neq m \in M\}$$

and replacing $\{m \in M : \{x, m\}_M = (\lambda + in)m \text{ for some } n \in \mathbb{Z}\}$ by $\{m \in M : \{u, m\}_M = (\lambda + 2n)m \text{ for some } n \in \mathbb{Z}\}$. □

Theorem 4.3.8. Let M be a finite-dimensional simple Poisson module annihilated by J_2. Let $d = \dim_{\mathbb{C}} M$. Then M has a basis $m_1, m_2, \ldots, m_d$ such that

(i) $\{u, m_j\}_M = (\lambda + 2(j-1))m_j$ for $1 \leq j \leq d$,

(ii) $\{x, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1}$ for $1 \leq j \leq d$,

(iii) $\{y, m_j\}_M = m_{j+1}$ for $1 \leq j < d$ and $\{y, m_d\}_M = 0$,

where $\lambda = 1 - d$.

Proof. The proof is similar to the proof of Theorem 4.2.9 with the eigenvalues $\lambda, \lambda + 2, \ldots, \lambda + 2(d-1)$ rather than $\lambda, \lambda + i, \ldots, \lambda + 2(d-1)i$. □

As a consequence from Lemma 4.3.6 and Theorem 4.3.8 with the action of x, y, u, we obtain the same result with the action of x, y, z as follows.

Theorem 4.3.9. *Let* $d \geq 1$. *There is a unique* $d-$*dimensional simple Poisson module over* A, *annihilated by* J_2. *It has a basis* $m_1, m_2, \ldots, m_d$ *such that*

(i) $\{x, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1}$ *for* $1 \leq j \leq d$,

(ii) $\{y, m_j\}_M = m_{j+1}$ *for* $1 \leq j < d$ *and* $\{y, m_d\}_M = 0$,

(iii) $\{z, m_j\}_M = (j-1)(\lambda+j-2)m_{j-1} + (\lambda+2(j-1))m_j + m_{j+1}$ *for* $1 \le j < d$ *and* $\{z, m_d\}_M = (d-1)(\lambda+d-2)m_{d-1} + (\lambda+2(d-1))m_d$*,*

where $\lambda = 1-d$*.*

Proof. It is immediate from Lemma 4.3.3, Lemma 4.3.6 and Theorem 4.3.8. □

Here is an example.

Example 4.3.10. Let $A = \mathbb{C}[x, y, u]$ with the Poisson bracket as in Lemma 4.3.3. The matrices representing the action $\{x, -\}_M, \{y, -\}_M$ and $\{u, -\}_M$ on the 3-dimensional and 4-dimensional simple Poisson modules, with respect to the basis in Theorem 4.3.8, are shown below:

3-dimensional simple module:

$$\{x,-\}_M = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \ \{y,-\}_M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \{u,-\}_M = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4-dimensional simple module:

$$\{x,-\}_M = \begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \{y,-\}_M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $\{u,-\}_M = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

To classify the general forms of finite-dimensional simple Poisson modules annihilated by J_j where $j = 3, 4, 5$, we make use of Poisson automorphisms of A.

4.4 Poisson automorphisms

Let $(R, \{-,-\})$ be a Poisson algebra. We say that a $\mathbb{C}$-algebra automorphism $\theta : R \to R$ is a **Poisson automorphism** if for all $x, y \in R$, $\theta(\{x,y\}) = \{\theta(x), \theta(y)\}$.

Theorem 4.4.1. *Let R be a commutative Poisson algebra with a $\mathbb{C}$-algebra automorphism α. Let $r, s, t \in R$. If $\{\alpha(r), \alpha(s)\} = \alpha\{r,s\}$ and $\{\alpha(r), \alpha(t)\} = \alpha\{r,t\}$ then $\{\alpha(r), \alpha(s+t)\} = \alpha\{r, s+t\}$ and $\{\alpha(r), \alpha(st)\} = \alpha\{r, st\}$. Therefore if X is a set of generators of R and $\alpha(\{x,y\}) = \{\alpha(x), \alpha(y)\}$ for all $x,\ y \in X$ then α is a Poisson automorphism of R.*

Proof. Let α be the Poisson automorphism of R. For each $r,\ s,\ t \in R$,

$$\begin{aligned}\{\alpha(r), \alpha(s+t)\} &= \{\alpha(r), \alpha(s) + \alpha(t)\} \\ &= \{\alpha(r), \alpha(s)\} + \{\alpha(r), \alpha(t)\} \\ &= \alpha\{r,s\} + \alpha\{r,t\} = \alpha\{r, s+t\}\end{aligned}$$

and

$$\begin{aligned}\{\alpha(r), \alpha(st)\} &= \{\alpha(r), \alpha(s)\alpha(t)\} \\ &= \{\alpha(r), \alpha(s)\}\alpha(t) + \alpha(s)\{\alpha(r), \alpha(t)\} \\ &= \alpha\{r,s\}\alpha(t) + \alpha(s)\alpha\{r,t\} \\ &= \alpha(\{r,s\}t + s\{r,t\}) = \alpha\{r, st\}\end{aligned}$$

It follows that for each $x \in X$, $\{s \in R : \alpha(\{x,s\}) = \{\alpha(x), \alpha(s)\}\}$ is a subalgebra of R containing X and hence equal to R. Also $\{r \in R : \alpha(\{r,s\}) = \{\alpha(r), \alpha(s)\}$ for all $s \in S\}$ is a subalgebra of R containing X and hence equal to R. So $\alpha\{r,s\} = \{\alpha(r), \alpha(s)\}$ for all $r, s \in R$. □

Theorem 4.4.2. *Let A be a Poisson algebra, let α be a Poisson automorphism of A and let M be a Poisson module. Define $a.m = \alpha(a)m$ and $\{a,m\}_M^\alpha = \{\alpha(a), m\}_M$ for all $a \in A$ and $m \in M$. Then M is a Poisson module under $-.- : A \times M \to M$ and $\{-,-\}_M^\alpha : A \times M \to M$.*

Proof. It is well-known that M is an $A-$module under $-.-$ see [3], so it suffices to check axioms $(i),(ii)$ and (iii) from Definition 1.9.8. To do this, let $a,a'\in A$ and $m\in M$. Firstly,

$$\begin{aligned}\{a,a'.m\}_M^\alpha &= \{\alpha(a),a'.m\}_M=\{\alpha(a),\alpha(a')m\}_M\\ &= \{\alpha(a),\alpha(a')\}m+\alpha(a')\{\alpha(a),m\}_M\\ &= \alpha\{a,a'\}m+\alpha(a')\{a,m\}_M^\alpha\\ &= \{a,a'\}.m+a'.\{a,m\}_M^\alpha.\end{aligned}$$

Secondly, we observe that

$$\begin{aligned}\{aa',m\}_M^\alpha &= \{\alpha(aa'),m\}_M=\{\alpha(a)\alpha(a'),m\}_M\\ &= \alpha(a)\{\alpha(a'),m\}_M+\alpha(a')\{\alpha(a),m\}_M\\ &= a.\{\alpha(a'),m\}_M+a'.\{\alpha(a),m\}_M\\ &= a.\{a',m\}_M^\alpha+a'.\{a,m\}_M^\alpha.\end{aligned}$$

Finally,

$$\begin{aligned}\{\{a,a'\},m\}_M^\alpha &= \{\alpha\{a,a'\},m\}_M=\{\{\alpha(a),\alpha(a')\},m\}_M\\ &= \{\alpha(a),\{\alpha(a'),m\}_M\}_M-\{\alpha(a'),\{\alpha(a),m\}_M\}_M\\ &= \{a,\{a',m\}_M^\alpha\}_M^\alpha-\{a',\{a,m\}_M^\alpha\}_M^\alpha.\end{aligned}$$

Therefore M is a Poisson A-module by Definition 1.9.8. □

Remark 4.4.3. Let M be a module.

(i) We denote the module M constructed in Theorem 4.4.2 by M^α.

(ii) The annihilator of M^α, $\operatorname{ann}_A M^\alpha=\alpha^{-1}(\operatorname{ann}_A M)$.

(iii) The Poisson submodules of M^α have the form N^α where N is a Poisson submodule of M.

(iv) A module M^α is a simple Poisson module if and only if M is a simple Poisson module.

In Example 4.1.3, the Poisson bracket of A is

$$\{x, y\} = yx + z, \quad \{y, z\} = zy + x, \quad \{z, x\} = xz + y.$$

Let α, β and γ be the $\mathbb{C}$-automorphisms of A such that

(i) $\alpha(x) = x,\ \alpha(y) = -y,\ \alpha(z) = -z,$

(ii) $\beta(x) = -x,\ \beta(y) = y,\ \beta(z) = -z,$

(iii) $\gamma(x) = -x,\ \gamma(y) = -y,\ \gamma(z) = z.$

Then we can check, using Theorem 4.4.1, that α, β and γ are Poisson automorphisms of R. Observe that $\alpha(J_2) = J_3$, $\beta(J_2) = J_4$ and $\gamma(J_2) = J_5$. As $\alpha^2 = \beta^2 = \gamma^2 = \mathrm{id}$, the simple Poisson modules annihilated by J_3 are precisely the Poisson modules M^α where M is a simple Poisson module annihilated by J_2. From Section 4.3, we can conclude that for each $d \geq 1$ there is precisely one $d-$dimensional simple Poisson module annihilated by J_3.

The simple Poisson modules annihilated by J_4 and J_5 are precisely the Poisson modules M^β and M^γ respectively, where M is a simple Poisson module annihilated by J_2. Again from Section 4.3, we conclude that for each $d \geq 1$ there is exactly one $d-$dimensional simple Poisson module annihilated by J_4 and J_5 respectively.

Combining the above remarks with the results of Section 4.2 and Section 4.3, we have the following result which can be compared with Theorem 3.9.6.

Theorem 4.4.4. *For $d \geq 1$, the Poisson algebra A has precisely five $d-$ dimensional simple Poisson modules.*

Proof. This is immediate from Theorem 4.2.11, Theorem 4.3.9 and using the Poisson automorphisms of A. □

4.5 Poisson modules over the quantum algebra $U_q(sl_2)$

In this section we apply the methods used earlier in the chapter to the Poisson algebra arising from another quantum algebra. This is the quantized enveloping algebra, $U_q(sl_2)$, and we consider a presentation discovered by Ito, Terwilliger and Weng [23] rather than the usual presentation as given in Definition 1.3.13.

Example 4.5.1. Let $q \neq 1$. The quantized enveloping algebra $U_q(sl_2)$ has a presentation with generators $x^{\pm 1}$, y, z and relations $xx^{-1} = x^{-1}x = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \tag{4.12}$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \tag{4.13}$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \tag{4.14}$$

Alternatively, the relations may be written as

$$qxy - q^{-1}yx = q - q^{-1}, \tag{4.15}$$

$$qyz - q^{-1}zy = q - q^{-1}, \tag{4.16}$$

$$qzx - q^{-1}xz = q - q^{-1}. \tag{4.17}$$

If $q = 1$ then these become

$$xy - yx = 0, \tag{4.18}$$

$$yz - zy = 0, \tag{4.19}$$

$$zx - xz = 0. \tag{4.20}$$

Let T be the $\mathbb{C}$-algebra with the generators x, y, z and $t^{\pm 1}$ subject to the relations

$$xy - t^{-2}yx = 1 - t^{-2}, \tag{4.21}$$

$$yz - t^{-2}zy = 1 - t^{-2}, \tag{4.22}$$

$$zx - t^{-2}xz = 1 - t^{-2} \quad \text{and} \tag{4.23}$$

$$xt = tx, \quad yt = ty, \quad zt = tz, \quad tt^{-1} = 1 = t^{-1}t.$$

Thus t is a central element of T. Let $A := T/(t-1)T \simeq \mathbb{C}[x, y, z]$ be the commutative polynomial algebra. By Definition 1.9.12, there is an induced Poisson bracket on A. In T, $[x,y] = xy - yx = t^{-2}yx - yx + (1-t^{-2}) = (1-t^{-2})(1-yx)$ therefore, in A, $\{x,y\} = 2(1-yx)$. Similarly, we obtain

$$\{y,z\} = 2(1-zy), \quad \{z,x\} = 2(1-xz).$$

Each maximal ideal I of A has the form $(x-a,\ y-b,\ z-c)$ for some $a,\ b,\ c \in \mathbb{C}$ by Theorem 1.4.6. Such an ideal is a Poisson ideal if and only if $\{x,y\} \in I$, $\{y,z\} \in I$, and $\{z,x\} \in I$. This happens if and only if $1-ab = 1-bc = 1-ac = 0$ and for this to happen $a,\ b,\ c$ are all non-zero. Since $ab = bc = ac = 1$, it follows that $b = abc = a = abc = c$. Hence $a = b = c = \pm 1$. Therefore we get

(i) $a = b = c = 1$ or

(ii) $a = b = c = -1$

This means that there are two Poisson maximal ideals of A:

(i) $I_1 = (x-1)A + (y-1)A + (z-1)A$;

(ii) $I_2 = (x+1)A + (y+1)A + (z+1)A$.

We will begin with the Poisson maximal ideal I_1.

4.6 Poisson modules annihilated by I_1

Lemma 4.6.1. *Let M be a Poisson module annihilated by*

$$I_1 = (x-1)A + (y-1)A + (z-1)A$$

and let $m \in M$. Then we have:

(i) $xm = ym = zm = m$.

(ii) *(a)* $\{xy, m\}_M = \{y, m\}_M + \{x, m\}_M$;

(b) $\{yz, m\}_M = \{z, m\}_M + \{y, m\}_M$;

(c) $\{zx, m\}_M = \{x, m\}_M + \{z, m\}_M$.

(iii) *(a)* $\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M = -2\{y, m\}_M - 2\{x, m\}_M$;

(b) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = -2\{y, m\}_M - 2\{z, m\}_M$;

(c) $\{z, \{x, m\}_M\}_M - \{x, \{z, m\}_M\}_M = -2\{x, m\}_M - 2\{z, m\}_M$.

Proof. (i) It is easily checked that $xm = ym = zm = m$.

(ii) By (i) and Definition 1.9.8(ii), we have

(a) $\{xy, m\}_M = x\{y, m\}_M + y\{x, m\}_M = \{y, m\}_M + \{x, m\}_M$;

(b) $\{yz, m\}_M = y\{z, m\}_M + z\{y, m\}_M = \{z, m\}_M + \{y, m\}_M$;

(c) $\{zx, m\}_M = z\{x, m\}_M + x\{z, m\}_M = \{x, m\}_M + \{z, m\}_M$.

(iii) (a) By using (ii) and Definition 1.9.8(iii), we can prove that

$$\begin{aligned}\{x, \{y, m\}_M\}_M - \{y, \{x, m\}_M\}_M &= \{\{x, y\}, m\}_M = \{2(1 - yx), m\}_M \\ &= -2\{y, m\}_M - 2\{x, m\}_M.\end{aligned}$$

(b) and (c) are proved similarly.

$\square$

Lemma 4.6.2. *Let M be a Poisson module annihilated by I_1. Let $\lambda \in \mathbb{C}$ be such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. Then*

(i) $\{x, \{y, m\}_M\}_M = (\lambda - 2)\{y, m\}_M - 2\lambda m$;

(ii) $\{y, \{z, m\}_M\}_M - \{z, \{y, m\}_M\}_M = -2\{y, m\}_M - 2\{z, m\}_M$;

(iii) $\{x, \{z, m\}_M\}_M = (\lambda + 2)\{z, m\}_M + 2\lambda m$.

Proof. These are easily checked by Lemma 4.6.1(iii). $\square$

To simplify calculations, we shall replace the generators y and z by $u := x + y$ and $v := x + z$.

Lemma 4.6.3. *Let $A = \mathbb{C}[x, y, z]$ be equipped with the Poisson bracket*

$$\{x, y\} = 2(1 - yx),\ \{y, z\} = 2(1 - zy),\ \{z, x\} = 2(1 - xz).$$

If $u = x + y$ and $v = x + z$ then x, u, v generate A and the Poisson bracket is given by

(i) $\{x, u\} = 2(1 - ux + x^2)$*;*

(ii) $\{x, v\} = -2(1 - xv + x^2)$*;*

(iii) $\{u, v\} = -2(1 - x(2v + 2u - 3x) + vu)$*.*

Proof. As $u = x + y$ and $v = x + z$, we have $y = u - x$ and $z = v - x$, and thus x, u, v generate A.

(i) $\{x, u\} = \{x, y\} = 2(1 - yx) = 2(1 - (u - x)x) = 2(1 - ux + x^2)$.

(ii) $\{x, v\} = \{x, z\} = -2(1 - xz) = -2(1 - xv + x^2)$.

(iii)

$$\begin{aligned}
\{u, v\} &= \{y, x\} + \{x, z\} + \{y, z\} \\
&= -2(1 - xz - yx + zy) \\
&= -2(1 - x(z + y) + zy) \\
&= -2(1 - 2xv - 2xu + 3x^2 + vu).
\end{aligned}$$

□

Lemma 4.6.4. *Let M be a Poisson module annihilated by*

$$I_1 = (u - 2)A + (v - 2)A + (x - 1)A$$

and let $m \in M$. Then we have:

(i) $xm = m$ *and* $vm = um = 2m$*.*

(ii) *(a)* $\{xu, m\}_M = 2\{x, m\}_M + \{u, m\}_M$*;*

(b) $\{xv, m\}_M = 2\{x, m\}_M + \{v, m\}_M$ *;*

(c) $\{uv, m\}_M = 2\{u, m\}_M + 2\{v, m\}_M$ *.*

(iii) *(a)* $\{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M = -2\{u, m\}_M$ *;*

(b) $\{x, \{v, m\}_M\}_M - \{v, \{x, m\}_M\}_M = 2\{v, m\}_M$ *;*

(c) $\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = 4\{x, m\}_M$ *.*

Proof. (i) It is obvious.

(ii) By (i) and Definition 1.9.8(ii),

(a) $\{xu, m\}_M = x\{u, m\}_M + u\{x, m\}_M = \{u, m\}_M + 2\{x, m\}_M$.

(b) and (c) are similarly proved.

(iii) (a) By (ii) and Definition 1.9.8(iii), we obtain

$$\begin{aligned} \{x, \{u, m\}_M\}_M - \{u, \{x, m\}_M\}_M &= \{\{x, u\}, m\}_M = \{2(1 - ux + x^2), m\}_M \\ &= -2(2\{x, m\}_M + \{u, m\}_M) + 4\{x, m\}_M \\ &= -2\{u, m\}_M \end{aligned}$$

(b) and (c) are similarly proved. □

Lemma 4.6.5. *Let $u = x + y$ and $v = x + z$ (so that $z = v - x$ and $y = u - x$). Let M be a finite-dimensional simple Poisson module annihilated by I_1. Let $\lambda \in \mathbb{C}$ be such that $\{x, m\}_M = \lambda m$ for some $0 \neq m \in M$. Then*

(i) $\{x, \{u, m\}_M\}_M = (\lambda - 2)\{u, m\}_M$ *;*

(ii) $\{x, \{v, m\}_M\}_M = (\lambda + 2)\{v, m\}_M$ *;*

(iii) $\{u, \{v, m\}_M\}_M - \{v, \{u, m\}_M\}_M = 4\lambda m$ *.*

Proof. This is routine using Lemma 4.6.4(iii). □

Lemma 4.6.6. *Let $A = \mathbb{C}[x, u, v]$ with the Poisson bracket as in Lemma 4.6.3. Let $d \geq 1$. There is a d-dimensional Poisson A-module M, with the basis $\{m_1, \dots, m_d\}$, such that $(x - 1)M = (u - 2)M = (v - 2)M = 0$ and* *(i)* $\{x, m_j\}_M = (d - 2j + 1)m_j$ *for* $1 \le j \le d$*;*

(ii) $\{v, m_j\}_M = 4(j - 1)(j - d - 1)m_{j-1}$ *for* $1 \le j \le d$*;*

(iii) $\{u, m_j\}_M = m_{j+1}$ *for* $1 \le j < d$ *and* $\{u, m_d\}_M = 0$*.*

Proof. By Lemma 4.1.2, it is enough to show that Definition 1.9.8(i) and (iii) hold for $m = m_j$ and $(a, a') = (x, u), (y, u)$ or (x, y) for the brackets defined (i), (ii) and (iii) above. We then extend the Poisson action on M from $V := \mathbb{C}x + \mathbb{C}y + \mathbb{C}u$ to $\mathbb{C}[x, y, u]$ using Definition 1.9.8(ii). Then the conclusions of Lemma 4.6.4(i) and (ii) hold.

To show Definition 1.9.8(iii) holds for $m = m_j$ and (a, a') defined as above, we first show that

$$\{\{x, u\}, m_j\}_M = \{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M \text{ for } 1 \le j \le d.$$

In the case $1 \le j < d$, by Lemma 4.6.3, let us consider

$$\begin{aligned} \{\{x, u\}, m_j\}_M &= \{\{x, y\}, m_j\}_M = 2\{1 - ux + x^2, m_j\}_M \\ &= -2\{ux, m_j\}_M + 4\{x, m_j\}_M = -2m_{j+1} \end{aligned}$$

and

$$\begin{aligned} &\{x, \{u, m_j\}_M\}_M - \{u, \{x, m_j\}_M\}_M \\ &\quad = \{x, m_{j+1}\}_M - (d - 2j + 1)\{u, m_j\}_M \\ &\quad = (d - 2(j + 1) + 1)m_{j+1} - (d - 2j + 1)m_{j+1} \\ &\quad = -2m_{j+1} = \{\{x, u\}, m_j\}_M \end{aligned}$$

In the case $j = d$, observe that $\{\{x, u\}, m_d\}_M = -2\{u, m_d\}_M = 0$ and

$$\{x, \{u, m_d\}_M\}_M - \{u, \{x, m_d\}_M\}_M = -(1 - d)\{u, m_d\}_M = 0 = \{\{x, u\}, m_d\}_M.$$

Secondly, we show that

$$\{\{x, v\}, m_j\}_M = \{x, \{v, m_j\}_M\}_M - \{v, \{x, m_j\}_M\}_M \text{ for } 1 \le j \le d.$$

We consider the case $1 \leq j < d$, by Lemma 4.6.3,

$$\begin{aligned}
\{\{x,v\},m_j\}_M &= -2\{1-xv+x^2,m_j\}_M \\
&= 2\{xv,m_j\}_M - 4\{x,m\}_M = 2\{v,m_j\}_M \\
&= 4(j-1)(j-d-1)m_{j-1},
\end{aligned}$$

and

$$\begin{aligned}
&\{x,\{v,m_j\}_M\}_M - \{v,\{x,m_j\}_M\}_M \\
&= 4(j-1)(j-d-1)\{x,m_{j-1}\}_M - (d-2j+1)\{v,m_j\}_M \\
&= 4(j-1)(j-d-1)(d-2j+3)m_{j-1} \\
&\quad -4(d-2j+1)(j-1)(j-d-1)m_{j-1} \\
&= 8(j-1)(j-d-1)m_{j-1} = \{\{x,v\},m_j\}_M
\end{aligned}$$

In the case $j = d$, we find that $\{\{x,v\},m_d\}_M = 2\{v,m_d\}_M = -8(d-1)m_{d-1}$ and

$$\begin{aligned}
&\{x,\{v,m_d\}_M\}_M - \{v,\{x,m_d\}_M\}_M \\
&= -4(d-1)\{x,m_{d-1}\}_M - (1-d)\{v,m_d\}_M \\
&= -4(d-1)(3-d)m_{d-1} - 4(d-1)^2 m_{d-1} \\
&= -8(d-1)m_{d-1} = \{\{x,v\},m_d\}_M.
\end{aligned}$$

Finally, we check that

$$\{\{u,v\},m_j\}_M = \{u,\{v,m_j\}_M\}_M - \{v,\{u,m_j\}_M\}_M \text{ for } 1 \leq j \leq d.$$

Now let us consider the case $1 \leq j < d$, by Lemma 4.6.3,

$$\begin{aligned}
\{\{u,v\},m_j\}_M &= -2\{1-2xv-2xu+3x^2+uv,m_j\}_M \\
&= 4\{xv,m_j\}_M + 4\{xu,m_j\}_M - 6\{x^2,m_j\}_M - 2\{uv,m_j\}_M \\
&= 4(2\{x,m_j\}_M + \{v,m_j\}_M) + 4(2\{x,m_j\}_M + \{u,m_j\}_M) \\
&\quad -12\{x,m_j\}_M - 2(2\{u,m_j\}_M + 2\{v,m_j\}_M) \\
&= -4\{x,m_j\}_M = 4(d-2j+1)m_j
\end{aligned}$$

and

$$\begin{aligned}
&\{u,\{v,m_j\}_M\}_M - \{v,\{u,m_j\}_M\}_M \\
&\qquad = 4(j-1)(j-d-1)\{u,m_{j-1}\}_M - \{v,m_{j+1}\}_M \\
&\qquad = 4(j-1)(j-d-1)m_j - 4j(j-d)m_j \\
&\qquad = 4(d-2j+1)m_j = \{\{u,v\},m_j\}_M.
\end{aligned}$$

In the case $j = d$, we see that $\{\{u,v\},m_d\}_M = 4\{x,m_d\}_M = 4(1-d)m_d$, and

$$\begin{aligned}
\{u,\{v,m_d\}_M\}_M - \{v,\{u,m_d\}_M\}_M &= -4(d-1)\{u,m_{d-1}\}_M \\
&= 4(1-d)m_d = \{\{u,v\},m_d\}_M.
\end{aligned}$$

It remains to show that Definition 1.9.8(i) holds for m and (a,a') as defined above.

We consider the case, $1 \leq j < d$. By Lemma 4.6.4(i), we first check that

$$\{x,um_j\}_M = 2\{x,m_j\}_M = 2(d-2j+1)m_j$$

and

$$\begin{aligned}
\{x,u\}m_j + u\{x,m_j\}_M &= 2(1-ux+x^2)m_j + 2\{x,m_j\}_M \\
&= 2\{x,m_j\}_M = 2(d-2j+1)m_j \\
&= \{x,um_j\}_M.
\end{aligned}$$

Secondly, $\{x,vm_j\}_M = 2\{x,m_j\}_M = 2(d-2j+1)m_j$ and

$$\begin{aligned}
\{x,v\}m_j + v\{x,m_j\}_M &= -2(1-xv+x^2)m_j + 2\{x,m_j\}_M \\
&= 2\{x,m_j\}_M = 2(d-2j+1)m_j \\
&= \{x,vm_j\}_M.
\end{aligned}$$

Finally, $\{u,vm_j\}_M = 2\{u,m_j\}_M = 2m_{j+1}$ and

$$\begin{aligned}
\{u,v\}m_j + v\{u,m_j\}_M &= -2(1-2xv-2xu+3x^2+vu)m_j + 2\{u,m_j\}_M \\
&= 2\{u,m_j\}_M = 2m_{j+1} = \{x,um_j\}_M.
\end{aligned}$$

Case: $j = d$. By Lemma 4.6.4(i), we first check that

$$\{x, um_d\}_M = 2\{x, m_d\}_M = (1-d)m_d$$

and

$$\{x, u\}m_d + u\{x, m_d\}_M = 2\{x, m_d\}_M = 2(1-d)m_d = \{x, um_d\}_M.$$

Secondly, $\{x, vm_d\}_M = 2\{x, m_d\}_M = 2(1-d)m_d$ and

$$\{x, v\}m_d + v\{x, m_d\}_M = 2\{x, m_d\}_M = 2(1-d)m_d = \{x, vm_d\}_M.$$

Finally, $\{u, vm_d\}_M = 2\{u, m_d\}_M = 0$ and

$$\{u, v\}m_d + v\{u, m_d\}_M = 0 = \{x, um_d\}_M.$$

This completes the proof. □

The d-dimensional Poisson modules constructed in Lemma 4.6.6 is simple. The proof is similar to the proof of the d-dimensional simple Poisson modules as in Section 4.2.

We now aim to show that the simple d-dimensional Poisson simple constructed above is unique.

Lemma 4.6.7. *Let M be a finite-dimensional simple Poisson module annihilated by I_1 and let $n \leq \dim_{\mathbb{C}} M$. Then there exist $\lambda \in \mathbb{C}$ and n linearly independent elements $m_1, m_2, \ldots, m_n \in M$ such that*

(i) $\{x, m_j\}_M = (\lambda - 2(j-1))m_j$ *for* $1 \leq j \leq n$*;*

(ii) $\{u, m_j\}_M = m_{j+1}$ *for* $1 \leq j < n$*;*

(iii) $\{v, m_1\}_M = 0$ *and* $\{v, m_j\}_M = 4(j-1)((j-2)-\lambda)m_{j-1}$ *for* $1 < j \leq n$*.*

Proof. The proof is similar to the proof of Lemma 4.2.8, using the generators x, u v, defining $\Lambda = \{\lambda \in \mathbb{C} : \{x, m\}_M = \lambda m \text{ for some } 0 \neq m \in M\}$ and replacing $\{m \in M : \{x, m\}_M = (\lambda + in)m \text{ for some } n \in \mathbb{Z}\}$ by

$$\{m \in M : \{x, m\}_M = (\lambda - 2n)m \text{ for some } n \in \mathbb{Z}\}.$$

□

Theorem 4.6.8. *Let M be a finite-dimensional simple Poisson module annihilated by I_1. Let $d = \dim_{\mathbb{C}} M$. Then M has a basis $m_1, m_2, \ldots, m_d$ such that*

(i) $\{x, m_j\}_M = (\lambda - 2(j-1))m_j$ *for* $1 \leq j \leq d$;

(ii) $\{v, m_1\}_M = 0$ *and* $\{v, m_j\}_M = 4(j-1)((j-2)-\lambda)m_{j-1}$ *for* $1 < j \leq d$;

(iii) $\{u, m_j\}_M = m_{j+1}$ *for* $1 \leq j < d$ *and* $\{u, m_d\}_M = 0$,

where $\lambda = d - 1$.

Proof. By Theorem 4.2.9 substitute to the sum of the eigenspaces for $\{x, -\}_M$ and the eigenvalues $\lambda, \lambda-2, \lambda-4, \ldots, \lambda-2(d-1)$. □

As a consequence from Lemma 4.6.6 and Theorem 4.6.8 with the action of x, y, u, we obtain the same result with the action of x, y, z as follows.

Theorem 4.6.9. *Let $d \geq 1$. There is a unique $d-$dimensional simple Poisson module over A, annihilated by I_1. It has a basis $m_1, m_2, \ldots, m_d$ such that*

(i) $\{x, m_j\}_M = (\lambda - 2(j-1))m_j$ *for* $1 \leq j \leq d$;

(ii) $\{y, m_j\}_M = m_{j+1} - (d - 2j + 1)m_j$ *for* $1 \leq j < d$ *and*
$\{y, m_d\}_M = (d-1)m_d$,

(iii) $\{z, m_j\}_M = 4(j-1)(j-d-1)m_{j-1} - (d-2j+1)m_j$ *for* $1 \leq j \leq d$.

where $\lambda = d - 1$.

Proof. This is immediate from Lemma 4.6.3, Lemma 4.6.6 and Theorem 4.6.8. □

Here are two examples.

Example 4.6.10. Let $A = \mathbb{C}[x, u, v]$ with the Poisson bracket as in Lemma 4.6.3. The matrices representing the action $\{x, -\}_M, \{u, -\}_M$ and $\{v, -\}_M$ on the 3-dimensional and 4-dimensional simple Poisson modules, with respect to the basis in Theorem 4.6.8, are shown below:

3-dimensional simple Poisson module:

$$\{x, -\}_M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \ \{u, -\}_M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \{v, -\}_M = \begin{pmatrix} 0 & -8 & 0 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

and 4-dimensional simple Poisson module:

$$\{x, -\}_M = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \{u, -\}_M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\{v, -\}_M = \begin{pmatrix} 0 & -12 & 0 & 0 \\ 0 & 0 & -16 & 0 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To classify a simple Poisson module annihilated by I_2, we make use of a $\mathbb{C}$-algebra automorphism in Theorem 4.4.1. Let α be the $\mathbb{C}$-automorphism of A such that

$$\alpha(x) = -x, \quad \alpha(y) = -y, \quad \alpha(z) = -z.$$

Then α is a Poisson automorphism of A become

$$\{\alpha(x), \alpha(y)\} = \{-x, -y\} = 2(1 - yx) = \alpha\{x, y\}.$$

Similarly, $\{\alpha(y), \alpha(z)\} = \alpha\{y, z\}$ and $\{\alpha(z), \alpha(x)\}$. Note that $\alpha(I_1) = I_2 = (x + 1)A + (y + 1)A + (z + 1)A$. The finite-dimensional simple Poisson modules annihilated by I_2 are precisely the modules M^α where M is finite-dimensional simple Poisson modules annihilated by I_1. Hence there are two d-dimensional simple Poisson modules for each $d \geq 1$.

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